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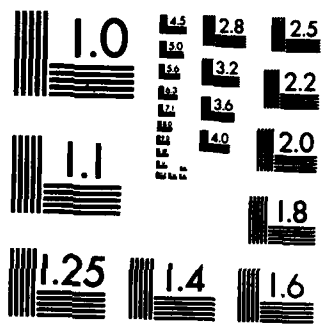


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Zonally asymmetric large scale atmospheric fields can be induced by local forcing effects such as topography and heat sources. The asymmetric effects are particularly strong when the group velocity of the long Rossby waves vanishes. This property is utilized in the framework of the two layer model to develop finite amplitude three dimensional solutions which possess zonally concentrated enhanced shear regions. Such flow fields can become locally unstable. The energy source for the instability can be baroclinic and/or barotropic. The stability investigation is numerical and a few preliminary results are presented.

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## 1. Introduction.

The discovery of baroclinic instability by Charney (1947) has dominated research in meteorology for the last thirty years. The large scale atmospheric flow has been regarded as a baroclinically unstable circum-polar vortex, the evolution of which is sought in the presence of transient disturbances. While it has been recognized that the mean flow of the atmosphere is asymmetric as a result of the zonal distribution of orography and heat sources, most of the studies assumed the mean flow to be zonally homogeneous. General circulation models took into account the asymmetric nature of the boundary terms but they were not very successful in improving our basic understanding of the consequences arising from the asymmetry. Yet such general circulation models did indicate that topography, for example, profoundly influences the behavior of the atmosphere (Manabe and Terpstra, 1974) as already had been known observationally for many years (Petterssen, 1956).

There are two notable dynamical features which are believed to be related to the presence of topographic and thermal asymmetries. They are the phenomenon of blocking and the observed geographical distribution of cyclone occurrence. Blocking refers to persistent large scale flow anomolies which tend to occur in certain geographical locations. Such a phenomenon is depicted in Figure 1 which shows that the high index westerly flow is blocked completely over the Atlantic in favor of large meridional excursions.

Although the blocking phenomenon has been known for many years (Rex 1950 a,b) only recently it has become the focal point of intense research activity. Tung and Lindzen (1979) suggested that atmospheric blocking could be explained in terms of simple linear resonance of planetary scale waves with respect to surface forcing such as continental elevation and land-sea differential heating. Egger (1978) proposed that

blocking could be the manifestation of barotropic nonlinear interaction among forced and slowly moving free waves. Charney and Devore (1979) suggested, using a highly truncated spectral model that blocking could be one possible quasi-stable equilibrium state of the atmosphere. Kalnay and Merkine (1981) performed numerical simulations with quasi-geostrophic barotropic flows in an open channel. The results revealed that when the system was repeatedly excited at some upstream location by localized disturbances an effectively time independent response occasionally emerged. The non-linear interaction of this field with a pre-existing steady asymmetric flow that was generated, for example, by localized topography or by potential vorticity sources led to new steady state configurations including blocking. Motivated by this work Merkine (1980, 1981) studied analytically the phenomenon of zero group velocity resonance of Rossby waves and obtained results in agreement with the work of Kalnay and Merkine (1981). In particular, he showed that the resonance phenomenon can generate intense currents which possessed closed circulations in agreement with the split jet configuration of some blocking events.

The possible relation between the blocking phenomenon and synoptic scale transients is implied in the barotropic model of Kalnay and Merkine (1981). Recently Illari and Marshal (1983) have also implicated the inhomogeneous eddy fluxes arising from synoptic scale transients in the maintenance of blocking patterns. If such connection exists then it is important to understand what determines the geographical distribution of cyclone occurence. But there are other and more obvious reasons for it, namely the improvement of local forecasts in regions of intense cyclogenetic activity such as the western Mediterranean and better understanding of the physical factors that determine the shift in the storm tracks.

The problem of understanding local instabilities is quite difficult. The localization of the phenomenon requires dealing with a spectrum of waves simultaneously and at the same time taking into consideration the fact that the basic flow is zonally varying. However, if we temporarily ignore the zonal variability of the basic flow but accept the fact that baroclinicity is more enhanced in certain regions then the concept of absolute instability (Merkine 1977, Merkine and Shafranek 1980), which is discussed in more detail later, may be applicable. Indeed, Pierrehumbert (1983) considered, in the context of the two layer model, the simplified problem of the evolution of baroclinic instability with infinite meridional scale in a zonally varying basic flow and showed that the eigenvalue of the local instability is determined by the constraint of absolute instability evaluated at the location of maximum baroclinicity. The stability of the zonally non-homogeneous atmospheric fields of January and July of the Northern Hemisphere was investigated numerically by Fredriksen (1983) and considerable agreement was found between the observed geographical distribution of the synoptic scale eddy heat flux and the distribution obtained from his numerical study. However, basic understanding of the underlying physics is still lacking. The various theoretical aspects related to Fredriksen's work are discussed most thoroughly by Pierrehumbert (1983).

The fundamental difficulty associated with stability analysis of zonally non homogeneous basic states is the non-separability of the zonal coordinate in the equations governing the linear evolution of the perturbation. However, the same difficulty arises at an earlier stage, namely in the determination of the basic state. Consequently only a few basic state analytical solutions of the non-linear equations can be found in the literature. When the forcing and the velocity field at infinity are assumed to be independent of the meridional direction the non-linear terms of the quasi-geostrophic equations vanish identically

and finite amplitude solutions which are independent of the meridional direction can be found by solving linear equations. This property was utilized by Merkine (1982) who studied the stability of such fields. The results were novel. They showed the existence of zero group velocity instability whereby an unstable non-propagating wave packet developed. The unstable wavepacket pivots about long zero group-velocity waves which extract energy from the non-homogeneous basic state but are unable to radiate it to infinity. The mechanism is similar to resonant triad interactions with one forced component. The instability is stronger and zonally more localized when the zonal non-homogeneity is stronger hence demonstrating in a unique way the subtle influence exerted by zonal non-homogeneous effects on the dynamics of quasi-geostrophic systems.

Deriving analytically finite amplitude steady state solutions which vary zonally and meridionally is very difficult. Nevertheless, when the conditions for zero group velocity resonance are satisfied simple analytical solutions can be found. The barotropic rectified currents of Merkine (1980, 1981) is one such example. The currents resemble typical atmospheric flow patterns with strong meridional variability and weak zonal dependence. These currents proved very useful for stability studies. The investigations of Merkine and Balgovind (1983) and Merkine (1983) reveal the existence of unstable localized barotropic wavepackets whose spatial structure and eigenfrequencies depend on two parameters which measure the degree of supercriticality (the strength of the horizontal shear) and the zonal length-scale of the shear region. The results indicate that the structure of the instability is determined by conditions that ensure the decay of the wavepacket at infinity and the transition from long to short waves across a turning point (critical layer) region which is controlled by non-parallel effects. The controlling influence exerted by the weak non-parallel effects on the evolution of the instability demonstrated the weakness of the

basic state zonal homogeneity assumption which can be used, away from critical layers, as a diagnostic tool only.

The stability study of Merkine and Balgovind (1983) dealt with barotropic fields. This is a simplifying assumption. The atmosphere and oceans are highly baroclinic as a result of the differential solar heating. The consequence is two-fold: the basic state is three-dimensional and the energetics of the instabilities is controlled by baroclinic and barotropic processes which are of comparable importance. The determination of three-dimensional basic states is naturally more complicated than in the corresponding barotropic problem and one approach is to study the observed mean fields of the atmosphere. This is the approach taken by Fredriksen (1983) who was able to obtain realistic results. The difficulty with such an approach is that the basic state is rather complicated and there is no way to determine the influence of various dynamical factors on the evolution of the instability.

The approach described in this report is different. We extend Merkine's (1980, 1981) resonance studies to two layer baroclinic systems and derive analytically finite amplitude three-dimensional fields. The horizontal shear, vertical shear and zonal variability of these solutions are controlled independently. This gives us much flexibility in assessing separately the influence of barotropic and baroclinic processes on the evolution of trapped instabilities in zonally varying mean flows. The two-layer model and some of its properties are described in Section 2. The three-dimensional steady state solutions are described in Section 3. The general stability problem is formulated in Section 4, while Section 5 formulates the linear stability problem. The necessary conditions for instability based on the parallel flow assumption are stated in Section 6 and applied to our particular basic state. The numerical model is described in Section 7 and some preliminary results are presented. Section 8 outlines the anticipated future progress of the research.

## 2. The model.

We consider the two-layer model for a slightly viscous quasi-geostrophic flow on a beta plane as described by Pedlosky (1970). Our particular system consists of two layers of homogeneous, immiscible fluid, with equal undisturbed depths confined vertically (in the z-direction) and meridionally (in the y-direction) by rigid horizontal boundaries. The system is unrestricted zonally (in the x-direction). The fluid density of the upper layer is slightly less than that of the lower layer so that the Boussinesq approximation can be invoked. The system rotates about the vertical axis with an angular velocity which is a linear function of the meridional direction as implied by the beta-plane approximation. Centrifugal effects are assumed negligible so that in the absence of relative motion the fluid interface is approximately level. Consistent with the quasi-geostrophic formalism viscous effects are confined to thin boundary layers adjacent to the rigid boundaries. The interface is assumed inviscid and surface tension effects are ignored. We denote upper layer fields by the index 1 and lower layer fields by the index 2 and decompose the total streamfunction $\Psi_n^*$ in the following way:

$$\begin{aligned} \Psi_1^* &= -(1+\tfrac{1}{2}\varepsilon)U^* y^* + (L^2/\tau)\psi_1^* \\ \Psi_2^* &= -(1-\tfrac{1}{2}\varepsilon)U^* y^* + (L^2/\tau)\psi_2^*. \end{aligned} \tag{2.1}$$

The asterisk denotes dimensional variables. The first term on the R.H.S of (2.1) describes a vertically sheared uniform flow. The characteristic velocity is $U^*$ and $\varepsilon$ is the non-dimensional shear parameter. The second term on the R.H.S of (2.1) describes the contribution to the total streamfunction arising from the presence of localized potential vorticity sources $S_n^*$ such as heat sources, for example, whose characteristic time scale is $\tau$. L is the dimensional distance between the meridional

walls. In the absence of sources the stationary flow field is everywhere westerly and $0 \leq \varepsilon \leq 2$.

The non-dimensional quasi-geostrophic equations governing $\psi_n$ are

$$\left(\frac{\partial}{\partial t} + (1 + \tfrac{1}{2}\varepsilon)\frac{\partial}{\partial x}\right)\left(\nabla^2\psi_1 + F(\psi_2 - \psi_1)\right) + (\beta + F\varepsilon)\frac{\partial\psi_1}{\partial x}$$

$$+ r\nabla^2\psi_1 = -\Delta J\left(\psi_1, \nabla^2\psi_1 + F(\psi_2 - \psi_1)\right) + S_1 \qquad (2.2)$$

$$\left(\frac{\partial}{\partial t} + (1 - \tfrac{1}{2}\varepsilon)\frac{\partial}{\partial x}\right)\left(\nabla^2\psi_2 + F(\psi_1 - \psi_2)\right) + (\beta - F\varepsilon)\frac{\partial\psi_2}{\partial x}$$

$$+ r\nabla^2\psi_2 = -\Delta J\left(\psi_2, \nabla^2\psi_2 + F(\psi_1 - \psi_2)\right) + S_2$$

where L and $U^*$ are the reference scales for non-dimensionalization. J is the Jacobian of two function and is defined as $J(f,g) = f_x g_y - f_y g_x$. In addition to $\varepsilon$ (2.2) contains the following non-dimensional parameters.

$\beta = \beta' L^2/U^*$

$r = (\nu f_0)^{\frac{1}{2}} L/DU^*$

$F = 2L^2 f_0^2/\left(gD(\rho_2 - \rho_1)/2\rho_2\right)$

$\Delta = L/U^*\tau$

where $f_0$ is the Coriolis parameter and $\beta'$ its gradient at the reference latitude. D is the depth of the system, $\nu$ is the kinematic viscosity, g is the gravitational acceleration and $\rho_n$ is the density. $\beta$ and F measure the importance of the beta effect and stratification, respectively. r measures the spin-down effect arising from the secondary circulation induced by the top and bottom Ekman layers. The meridional walls serve to restrict the lateral scale of the domain and it is immateri l whether they are considered as viscous or inviscid since they do not induce a secondary interior circulation. (Greenspan, 1968). $\Delta$ is the parameter of non-linearity. It measures the strength of the potential vorticity sources. When $\Delta \to 0$ the

potential vorticity sources are very weak and non-linear effects can be ignored.

It is convenient to separate barotropic and baroclinic effects. Consequently we write

$$\psi_m = \frac{1}{2}(\psi_1 + \psi_2) \quad ; \quad \psi_\tau = \frac{1}{2}(\psi_1 - \psi_2) \tag{2.3}$$

where $\psi_m$ and $\psi_\tau$ denote the barotropic and baroclinic modes, respectively. By adding and subtracting the two equations of (2.2) and using (2.3) we obtain the following two equations

$$\begin{gathered}
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\nabla^2\psi_m + \beta\,\frac{\partial\psi_m}{\partial x} + r\nabla^2\psi_m + \frac{1}{2}\,\varepsilon\,\frac{\partial}{\partial x}\,\nabla^2\psi_\tau \\
= -\,\Delta J(\psi_m,\nabla^2\psi_m) - \Delta J(\psi_\tau,\nabla^2\psi_\tau) + S_m \\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\nabla^2\psi_\tau - 2F\psi_\tau\right) + \beta\,\frac{\partial\psi_\tau}{\partial x} + r\nabla^2\psi_\tau + \frac{1}{2}\,\varepsilon\,\frac{\partial}{\partial x}\,\nabla^2\psi_m \\
+ F\,\varepsilon\,\frac{\partial}{\partial x}\,\psi_m = -\Delta J(\psi_m,\nabla^2\psi_\tau) - \Delta J(\psi_\tau,\nabla^2\psi_m) \\
+ 2F\Delta J(\psi_m,\psi_\tau) + S_\tau
\end{gathered} \tag{2.4}$$

where

$$S_m = \frac{1}{2}(S_1 + S_2) \quad , \quad S_\tau = \frac{1}{2}(S_1 - S_2) \tag{2.5}$$

are the respective barotropic and baroclinic components of the potential vorticity sources. Equations (2.4) were used by Merkine (1982) to study the stability of fields induced by localized potential vorticity sources which vary in the zonal direction only. Our goal is to study the stability properties of more general fields.

Inspection of (2.4) reveals that the vertical shear of the uniform current which is measured by $\varepsilon$ and the non-linear interaction terms which are measured by $\Delta$ are the two physical entities which couple the dynamics of the barotropic and baroclinic modes of motion. When non-linear effects can be ignored (2.4) reduces to

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\nabla^2\psi_m + \beta\frac{\partial\psi_m}{\partial x} + r\nabla^2\psi_m + \frac{1}{2}\varepsilon\frac{\partial}{\partial x}\nabla^2\psi_\tau = S_m$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\nabla^2\psi_\tau - 2F\psi_\tau\right) + \beta\frac{\partial\psi_\tau}{\partial x} + r\nabla^2\psi_\tau + \frac{1}{2}\varepsilon\frac{\partial}{\partial x}\nabla^2\psi_m + F\varepsilon\frac{\partial}{\partial x}\psi_m = S_\tau \quad (2.6)$$

and when the basic uniform current possesses no vertical shear (2.6) reduces to

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\nabla^2\psi_m + \beta\frac{\partial\psi_m}{\partial x} + r\nabla^2\psi_m = S_m$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\nabla^2\psi_\tau - 2F\psi_\tau\right) + \beta\frac{\partial\psi_\tau}{\partial x} + r\nabla^2\psi_\tau = S_\tau. \quad (2.7)$$

The reason for the nomenclature $\psi_m$ and $\psi_\tau$ is clear now. ie free modes of the first equation of (2.7) are barotropic ssby waves while those of the second equation of (2.7) are ba clinic Rossby waves. Since

$$\psi_1 = \psi_m + \psi_\tau \quad , \quad \psi_2 = \psi_m - \psi_\tau \quad (2.8)$$

(2.7) implies that $\psi_1 = \psi_2$ for $S_\tau = 0$ and $\psi_1 = -\psi_2$ for $S_m = 0$. When small vertical shear is present $\psi_m$ and $\psi_\tau$ are coupled: The barotropic mode is modified by baroclinic effects and the baroclinic mode is modified by barotropic effects. For barotropic forcing $S_\tau = 0$ and (2.6) indicates that $\psi_\tau = 0(\varepsilon)$ and that the baroclinic modification of $\psi_m$ is $0(\varepsilon^2)$. When $S_m = 0$ $\psi_m = 0(\varepsilon)$ and the barotropic modification of $\psi_\tau$ is $0(\varepsilon^2)$. When $\varepsilon$ is no longer small the coupling between $\psi_m$ and $\psi_\tau$ is strong regardless of the nature of the forcing and the identification of $\psi_m$ and $\psi_\tau$ with separate barotropic and baroclinic dynamical responses is not meaningful anymore. Nevertheless, we still find it convenient to refer to $\psi_m$ and $\psi_\tau$ as the barotropic and baroclinic modes of motion since they assume their pure form when $\varepsilon \to 0$. When $\varepsilon = 0$ and $\Delta \neq 0$ the barotropic and baroclinic modes interact non-linearly (see (2.4)) but the spectrum can be separated into pure barotropic and baroclinic modes. We seek

We seek solutions to (2.2) which result from localized forcing and initial conditions. Consequently we assume that $\psi_n$ decays to zero at infinity implying that the condition of vanishing meridional velocity on the meridional boundaries can be stated as

$$\psi_1 = \psi_2 = \psi_m = \psi_\tau = 0 \quad \text{on} \quad y = 0,1. \tag{2.9}$$

The dynamical nature of the response of the system depends crucially on the characteristics of the free waves. Hence, we seek solutions of the form

$$\begin{pmatrix} \psi_m \\ \psi_\tau \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+\gamma \\ 1-\gamma \end{pmatrix} e^{i(kx-\omega t)} \sin(m\pi y) + \text{c.c.}; \; m = 1,2,\ldots \tag{2.10}$$

where c.c. denotes complex conjugation and substitute (2.10) into (2.6) with $S_m = S_\tau = 0$ to obtain the following dispersion relation and vertical structure of the waves

$$\frac{\omega}{k} = c = 1 - \frac{K^2+F}{K^2+2F}\left(\frac{\beta}{K^2} + \frac{ir}{k}\right) \pm \frac{[\varepsilon^2 K^4(K^4-4F^2) + 4F^2(\beta+irK^2/k)^2]^{\frac{1}{2}}}{2K^2(K^2+2F)} \tag{2.11}$$

$$\gamma = \frac{K^2+F}{F} - \frac{\beta+F\varepsilon+irK^2/k}{(1+\frac{1}{2}\varepsilon - c)F} \tag{2.12}$$

where $K^2 = k^2+m^2\pi^2$. Equations (2.11) - (2.12) are identical to the corresponding equations of Pedlosky (1970).

In the atmosphere and oceans the radius of deformation $L_R = \left(gD(\rho_2-\rho_1)/(2\rho_2)\right)^{\frac{1}{2}}/f_0$ is comparable to $(U^*/\beta')^{\frac{1}{2}}$ which is the typical length of stationary Rossby waves and consequently we set $F = \beta$. Baroclinic instability sets in whenever the shear parameter exceeds a certain threshold value. When $r \equiv 0$ the neutral stability curve is given by

$$\varepsilon^2 = \frac{4\beta^4}{K^4(4\beta^2-K^4)} \quad (2.13)$$

and the critical value of the shear is $|\varepsilon| = 1$. When $r \neq 0$ the neutral stability curve is given by

$$\varepsilon^2 = \frac{4}{(2\beta-K^2)} \left[\frac{\beta^4}{K^2(K^2+\beta)^2} + \frac{K^2r^2}{k^2}\right] \quad (2.14)$$

which does not reduce to (2.13) when $r \to 0$. This singular behavior was observed first by Holopainen (1961) who also noticed that small r destabilizes the system. The threshold value of $\varepsilon$ derived from (2.14) is equal to $|\varepsilon| \simeq 0.908333 + O(r^2)$ which is less than the above critical value for the strictly inviscid system. In both cases, however, the spectrum of the unstable waves is restricted to $0 < K^2 < 2\beta$. The waves to be destabilized first when $r \equiv 0$ and $r \to 0^+$ are given by $K^2 = 2^{\frac{1}{2}}\beta$ and $K^2 = (1+3^{\frac{1}{2}})\beta/2 + O(r^2)$, respectively.

When the shear is subcritical and the frictional effects secondary in importance each branch of the dispersion relation resembles a cubic. Both branches are antisymmetric functions of k about $k = 0$. The number of real roots of $\omega = 0$ for each branch depends on the sign of $d\omega/dk$ $(k = 0)$. When it is positive we have one real root and when it is negative we have three real roots. When it vanishes we have the coalescence at the origin of the three real roots. This latter situation is of particular importance. When it occurs weak stationary forcing can trigger a large stationary response. $d\omega/dk$ is the group velocity of the waves and when it vanishes at the origin the energy pumped by the sources into the long waves does not escape to infinity and a strong response builds up. This is the phenomenon of zero group velocity resonance discussed by Merkine (1980, 1981) when he considered the response of a pure barotropic system.

## 3. Steady state solutions.

We return now to (2.4) and seek steady state solutions for time independent localized sources. The problem is non-linear and our approach is to assume that non-linear effects are small and to expand the field variables in the parameter of non-linearity $\Delta$. Consequently we write

$$\begin{aligned}\psi_m &= \psi_m^\circ + \Delta\psi_m' + \ldots \\ \psi_\tau &= \psi_\tau^\circ + \Delta\psi_\tau + \ldots ,\end{aligned} \tag{3.1}$$

substitute (3.1) into (2.4) and equate to zero the coefficients of the various powers of $\Delta$. At the lowest order we recover the linear problem, namely

$$\begin{aligned}&\frac{\partial}{\partial x}(\nabla^2\psi_m^\circ+\beta\psi_m^\circ) + r\nabla^2\psi_m^\circ + \frac{1}{2}\,\varepsilon\,\frac{\partial}{\partial x}\,\nabla^2\psi_\tau^\circ = S_m \\ &\frac{\partial}{\partial x}(\nabla^2\psi_\tau^\circ-\beta\psi_\tau^\circ) + r\nabla^2\psi_\tau^\circ + \frac{1}{2}\,\varepsilon\,\frac{\partial}{\partial x}(\nabla^2\psi_m^\circ+2\beta\varepsilon\psi_m^\circ) = S_t\end{aligned} \tag{3.2}$$

(Recall that $F = \beta$)

We follow Merkine (1980, 1981) and assume that the sources have the simplified form

$$\begin{aligned}S_m &= \mu\delta(x)\ \sin m\pi y \quad ; \quad \mu = 0, \pm 1 \\ S_\tau &= \nu\delta(x)\ \sin m\pi y \quad ; \quad \nu = 0, \pm 1\end{aligned} \tag{3.3}$$

where $\delta(x)$ is Dirac's delta function. This choice is not as restrictive as it seems to be. At zero group velocity resonance, which is the dynamically relevant case, a single meridional wavenumber dominates the response whose zonal structure consists of very long waves. Thus, had we considered a general source function $S_n = f_n(x,y)$ the end result would have been the same

but the algebra necessarily more complicated (Merkine 1980, 1981). $\mu$ and $\nu$ need not be equal.

The meridional structure of the forcing suggests that (3.2) admits solutions of the form

$$\psi_m^\circ = \varphi_m(x)\sin(m\pi y) \quad , \quad \psi_\tau^\circ = \varphi_\tau(x)\sin(m\pi y). \tag{3.3}$$

The equations that govern $\varphi_m(x)$ and $\varphi_\tau(x)$ are suitable for the application of the Fourier transform defined as

$$\hat{P}(k) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} P(x)e^{-ikx}dx$$

$$P(x) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \hat{P}(k)e^{ikx}dk \tag{3.4}$$

where the caret denotes the transformed variable. It follows that the solution for $\hat{\phi}_m$ and $\hat{\phi}_\tau$ is

$$\hat{\phi}_m(k) = \frac{i}{(2\pi)^{\frac{1}{2}}}\left\{\mu\left[k(k^2+m^2\pi^2+\beta) + ir(k^2+m^2\pi^2)\right]-\nu k\varepsilon(k^2+m^2\pi^2)/2\right\}/D$$

$$\hat{\phi}_\tau(k) = \frac{i}{(2\pi)^{\frac{1}{2}}}\left\{\mu k\varepsilon\left[\beta-(k^2+m^2\pi^2)/2\right]-\nu\left[k(\beta-k^2-m^2\pi^2) + ir(k^2+m^2\pi^2)\right]\right\}/D \tag{3.5}$$

where

$$\begin{aligned} D = {} & (1-\varepsilon^2/4)k^6 + \left(2(1-\varepsilon^2/4)m^2\pi^2 + \varepsilon^2\beta/2\right)k^4 \\ & + \left((1-\varepsilon^2/4)m^4\pi^4 - \beta(\beta-\varepsilon^2m^2\pi^2/2)\right)k^2 \\ & - (r^2 + 2irk)(k^2+m^2\pi^2)^2. \end{aligned} \tag{3.6}$$

The condition for zero group velocity of the long waves, namely $d\omega/dk\ (k = 0\ ;\ r = 0) = 0$ can be determined from (2.11). It can be shown, however, that it is equivalent to the condition

$$(1 - \tfrac{1}{4}\varepsilon^2)m^4\pi^4 - \beta(\beta - \varepsilon m^2\pi^2/2) = 0 \tag{3.7}$$

obtained by equating to zero the coefficient of $k^2$ in the third expression of (3.6). Solving for $\beta$ we obtain that

$$\beta = [\varepsilon^2 + (16 + \varepsilon^4 - 4\varepsilon^2)^{1/2}]m^2\pi^2/4, \tag{3.8}$$

which is valid also for super-critical shear since the long waves are always stable. When $\varepsilon \to 0$ , $\beta = m^2\pi^2$. This is the condition for zero group velocity of the long waves in the pure barotropic problem (Merkine, 1980, 1981). It follows that (3.8) is the condition for zero group velocity of the barotropic branch of the dispersion relation (2.11). Equation (3.7) admits a second solution for $\beta$ corresponding to the baroclinic mode of the dispersion relation. In our case it is negative implying that the long baroclinic waves cannot possess zero group velocity. This conclusion is restricted, however, to the case $\beta = F$ treated here. When $\beta \neq F$ the long baroclinic waves can have zero group velocity for sufficiently large $\varepsilon$ (Merkine, 1982).

The implication of (3.8) is that when $r = 0$ the inverse Fourier transform of (3.5) does not exist implying unbounded solutions. This follows from the fact that the inviscid system operates at resonance. The energy pumped by the source into the longwaves cannot be radiated to infinity or dissipated by frictional processes and no stationary solutions are possible. Stationary solutions exist when friction is included and we assume that $r << 1$ which is applicable for quasi-geostrophic atmospheric and oceanic systems.

The solution for $\varphi_m$ and $\varphi_\tau$ is determined by applying Cauchy's theorem to the inverse Fourier transform of (3.5). The six poles of the integrand are the zeros of (3.6) subject to (3.7). They are given by

$$k = \frac{1}{2}\, ir + o(r)$$

$$k = \frac{(2m^4\pi^4)^{1/3}\exp i\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right)}{[(2-\varepsilon^2/2)m^2\pi^2+\varepsilon^2\beta/2]}\, r^{1/3} + o(r^{1/3}) \;;\; n = 0,1,2 \tag{3.9}$$

$$k = \pm\, i[(2 - \varepsilon^2/2)m^2\pi^2 + \varepsilon^2\beta/2]^{1/2}/(1-\varepsilon^2/4)^{1/2} + o(1).$$

It follows that the solution for $\varphi_m$ and $\varphi_\tau$ is

$$\begin{pmatrix} \varphi_m \\ \\ \varphi_\tau \end{pmatrix} = -\frac{k_1}{6r^{2/3}m^4\pi^4}\begin{pmatrix} \mu(\beta + m^2\pi^2) - \nu\varepsilon m^2\pi^2/2 \\ \\ \mu\varepsilon(\beta - m^2\pi^2/2) - \nu(\beta - m^2\pi^2) \end{pmatrix} R(x)$$

$$R(x) = \begin{cases} \exp(r^{1/3}k_1 x) & , \quad x \le 0 \\ -2\exp(-r^{1/3}k_1 x/2)\cos(\frac{1}{2}3^{\frac{1}{2}}k_1 r^{1/3}x + \frac{2}{3}\pi) & , \quad x \ge 0 \end{cases} \qquad (3.10)$$

$$k_1 = \left[\frac{2m^4\pi^4}{(2-\varepsilon^2/2)m^2\pi^2 + \varepsilon^2\beta/2}\right]^{1/3}$$

$$\beta = [\varepsilon^2 + (16+\varepsilon^4-4\varepsilon^2)^{\frac{1}{2}}]m^2\pi^2/4.$$

For given $\varepsilon$ and m the resonance condition (3.8), which is also the last expression of (3.10) determines $\beta$ and the complete structure of the solution is known.

The dimensionless form of the total streamfunction (2.1) is

$$\begin{aligned} \Psi_1 &= -(1+\varepsilon/2)y + \Delta\psi_1 \\ \Psi_2 &= -(1-\varepsilon/2)y + \Delta\psi_2 \end{aligned} \qquad (3.11)$$

and with the aid of (2.8), (3.1), (3.3) and (3.10) it can be written as

$$\begin{pmatrix} \Psi_1 \\ \\ \Psi_2 \end{pmatrix} = -\begin{pmatrix} 1+\varepsilon/2 \\ \\ 1-\varepsilon/2 \end{pmatrix} y - \frac{\Delta}{r^{2/3}}\left\{\frac{k_1}{6m^4\pi^4}\begin{pmatrix} \mu[\beta(1+\varepsilon)+m^2\pi^2(1-\varepsilon/2)] - \nu[\beta - m^2\pi^2(1-\varepsilon/2)] \\ \\ \mu[\beta(1-\varepsilon)+m^2\pi^2(1+\varepsilon/2)] + \nu[\beta - m^2\pi^2(1+\varepsilon/2)] \end{pmatrix} \times \right. \qquad (3.12)$$

$$\left. R(x)\sin m\pi y + O(r^{2/3} + \Delta) \right\} .$$

The error estimate in (3.12) consists of two terms. The first one reflects inaccuracies due to the asymptotic estimate of the poles of the linear solution, i.e., eq. (3.9)*. The second term estimates the relative error due to nonlinear effects. The interesting situation occurs when $\Delta = 0(r^{2/3})$. Then, the field induced by the sources becomes comparable to the zonal flow and the relative error estimate is $0(r^{2/3})$ and hence small in an asymptotic sense. We have reached an important result. The solution of the linear equation (3.2) is in fact a very accurate finite amplitude solution of the full non-linear problem (2.4). This surprising result which rarely happens is a consequence of the fact that we operate at resonance conditions. In the presence of very weak forcing and dissipation the resonating solution is almost an eigenfucntion of the linear equations. In other words, it is a Rossby wave which is maintained by weak forcing against weak dissipation. But it is well known that a single finite amplitude Rossby wave satisfies exactly the non-linear equation and this explains the small relative error associated with the contribution of the non-linear terms to the solution (3.12).

The solution given by (3.12) describes a current possessing vertical and horizontal shear which varies slowly in the zonal direction. The length scale of the zonal variation is $0(r^{-1/3})$ and this non-parallel effect is accompanied by a small meridional velocity which is of $0(\Delta/r^{2/3})$ or $0(r^{1/3})$ when $\Delta = 0(r^{2/3})$. The varying part of the solution decays to zero at large distances from the region of the sources. The meridional structure of the current is very simple and it does not vary in the zonal

---

* Note that when the sources are of general type $0(\Delta)$ is the proper estimate of all contributions which are not directly related to resonance.

direction. When $\varepsilon = 0$ , $\beta = m^2\pi^2$, the resonance is pure barotropic and regardless of the value of $\nu$ (3.12) reduces to the solution derived by Merkine (1980) for the barotropic case. From (3.12) it follows that $\beta$ is a monotone function of $\varepsilon$. It is equal to $m^2\pi^2$, 1.15 $m^2\pi^2$ and 2 $m^2\pi^2$ for $\varepsilon = 0$, 1 and 2, respectively. For typical mid-latitude values of $U^* \simeq 10$ m/sec, $\varepsilon \simeq 1$, $L \simeq 5.10^6$ m and $\beta'$ evaluated at 45° the resonance condition is satisfied when $m = 2$. This result is comforting since the solution (3.12) with $m = 2$ corresponds to a slowly varying jet.

The derivation of (3.12) is based on the properties of the dispersion relation for long waves. Such waves are always stable according to the linear theory but is the solution (3.12) meaningful when $\varepsilon$ is supercritical such that a part of the spectrum is baroclinically unstable? The answer to this question is found in the concept of absolute instability introduced first into geophysical fluid dynamics by Merkine (1977). The way a medium responds to a given forcing depends on whether the linear instability is convective or absolute. When it is convective the baroclinically unstable transient set up by the initial value problem leaves the domain of its excitation and the long time response at a fixed spatial point attains the time dependent characteristics of the forcing. If the forcing is steady the response is steady. However, when the system supports absolute instability a part of the unstable transient never leaves the domain of its excitation and the long time response at any given spatial location is dominated by an exponentially growing solution. The growth rate is determined by the imaginary value of $\omega$ corresponding to the condition $d\omega/dk = 0$ in the complex k-plane. The existence of absolute instability in a given situation depends on the parameters of the problem. When we apply the resonance parameter constraint

to the criteria for absolute instability given by Merkine and Shafranek, (1980) we find that the steady state solution is physically realizable unless $\varepsilon \approx 2$.

The problem relating to the stability properties of (3.12) is a different issue. When certain conditions for instability are satisfied (3.12) may become unstable to perturbations. Since our steady state solution is three-dimensional there is a rich interplay between baroclinic and barotropic processes both influenced by zonal localization effects. The stability analysis of (3.12) under a variety of conditions is the main thrust of the study. We hope that the results obtained will contribute to the understanding of the difficult but important problem of the influence of zonal asymmetric effects such as topography and heat sources on the dynamics of the atmosphere and oceans.

## 4. The formulation of the stability problem.

It is convenient to start the stability analysis by writing (2.2) with $\beta = F$ in the conservation form

$$
\begin{aligned}
&\left[\frac{\partial}{\partial t} + \left(1+\tfrac{1}{2}\varepsilon - \frac{\partial \Delta\psi_1}{\partial y}\right)\frac{\partial}{\partial x} + \frac{\partial \Delta\psi_1}{\partial x}\,\frac{\partial}{\partial y}\right]\left[\nabla^2\Delta\psi_1 + \beta(\Delta\psi_2 - \Delta\psi_1) + \beta(1+\varepsilon)y\right] \\
&\qquad = -\,r\nabla^2\Delta\psi_1 + \Delta S_1 \\
&\left[\frac{\partial}{\partial t} + \left(1-\tfrac{1}{2}\varepsilon - \frac{\partial \Delta\psi_2}{\partial y}\right)\frac{\partial}{\partial x} + \frac{\partial \Delta\psi_2}{\partial x}\,\frac{\partial}{\partial y}\right]\left[\nabla^2\Delta\psi_2 + \beta(\Delta\psi_1 - \Delta\psi_2) + \beta(1-\varepsilon)y\right] \\
&\qquad = -\,r\nabla^2\Delta\psi_2 + \Delta S_2
\end{aligned}
\tag{4.1}
$$

which states that the rate of change of the quasi-geostrophic potential vorticity of each layer following the geostrophic motion of that layer is determined by the potential vorticity sources and the spin-down effect of the Ekman layers.

We add now a perturbation to the flow field such that

$$\Delta\psi_n \Rightarrow \Delta\psi_n + \delta\varphi_n$$

where $\delta$ measures the strength of the perturbation. When the latter expression is substituted into (4.1), while taking into account the fact that $\psi_n$ satisfies (2.4) to $O(r^{2/3})$ (as stated earlier the case of interest is when $\Delta = O(r^{1/3})$, we obtain the following system of equations that governs the evolution of the perturbation field $\varphi_n$.

$$
\begin{aligned}
&\left[\frac{\partial}{\partial t} + (1+\tfrac{1}{2}\varepsilon + U_1)\frac{\partial}{\partial x} + V_1\,\frac{\partial}{\partial y}\right]q_1 + \frac{\partial\varphi_1}{\partial x}\,\frac{\partial\pi_1}{\partial y} - \frac{\partial\varphi_1}{\partial y}\,\frac{\partial\pi_1}{\partial x} \\
&\qquad + \delta\left(\frac{\partial\varphi_1}{\partial x}\,\frac{\partial q_1}{\partial y} - \frac{\partial\varphi_1}{\partial y}\,\frac{\partial q_1}{\partial x}\right) = O(r^{2/3}) \\
&\left[\frac{\partial}{\partial t} + (1-\tfrac{1}{2}\varepsilon + U_2)\frac{\partial}{\partial x} + V_2\,\frac{\partial}{\partial y}\right]q_2 + \frac{\partial\varphi_2}{\partial x}\,\frac{\partial\pi_2}{\partial y} - \frac{\partial\varphi_2}{\partial y}\,\frac{\partial\pi_2}{\partial y} \\
&\qquad + \delta\left(\frac{\partial\varphi_2}{\partial x}\,\frac{\partial q_2}{\partial y} - \frac{\partial\varphi_2}{\partial y}\,\frac{\partial q_2}{\partial x}\right) = O(r^{2/3}).
\end{aligned}
\tag{4.2}
$$

$q_n$, n = 1,2 is the perturbation's potential vorticity defined as

$$q_1 = \nabla^2\varphi_1 + \beta(\varphi_2 - \varphi_1)$$
$$q_2 = \nabla^2\varphi_2 + \beta(\varphi_1 - \varphi_1) \tag{4.3}$$

and $\pi_n$, n = 1, 2 is the potential vorticity of the basic state (3.12) defined as

$$\pi_1 = \beta(1+\varepsilon)y + \frac{\partial^2\Delta\psi_1}{\partial y^2} + \beta(\psi_2-\psi_1) + O(r^{2/3})$$
$$\tag{4.4}$$
$$\pi_2 = \beta(1-\varepsilon)y + \frac{\partial^2\Delta\psi_2}{\partial y^2} + \beta(\psi_1-\psi_2) + O(r^{2/3}).$$

The geostrophic velocity induced by the sources is given by

$$U_n = -\frac{\partial\Delta\psi_n}{\partial y}, \quad V_n = \frac{\partial\Delta\psi_n}{\partial x} \qquad n = 1,2 \tag{4.5}$$

where $U_n = O(1)$ and $V_n = O(r^{1/3})$. Using the properties of the steady state solution (3.12) we can write (4.4) as

$$\pi_1 = \beta(1+\varepsilon)y - (m^2\pi^2+\beta)\Delta\psi_1 + \beta\Delta\psi_2 + O(r^{2/3})$$
$$\pi_2 = \beta(1-\varepsilon)y - (m^2\pi^2+\beta)\Delta\psi_2 + \beta\Delta\psi_1 + O(r^{2/3}) \tag{4.6}$$

the gradient of which is given by

$$\nabla\pi_1 = [-(m^2\pi^2+\beta)V_1+\beta V_2]\hat{i}$$
$$+ [\beta(1+\varepsilon)+(m^2\pi^2+\beta)U_1-\beta U_2]\hat{j} + O(r^{2/3})$$
$$\tag{4.7}$$
$$\nabla\pi_2 = [-(m^2\pi^2+\beta)V_2+\beta V_1]\hat{i}$$
$$+ [\beta(1-\varepsilon)+(m^2\pi^2+\beta)U_2-\beta U_1]\hat{j} + O(r^{2/3}).$$

Equations (3.2) and (3.3) supplemented by the above definitions of the basic state, the boundary conditions

$$\varphi_n = 0 \quad \text{on} \quad y = 0,1 \quad ; n = 1,2 \tag{4.8}$$

and suitable initial conditions constitute a well posed initial value problem which determines the evolution of the perturbation correct to $0(r^{1/3})$ in non-parallel effects.

## 5. The linear stability problem.

Our prime interest is in the evolution of the perturbation field under unstable conditions. When the amplitude of the growing perturbation is still small compared to the basic state, non-linear effects are of secondary importance. The most logical and certainly the most common starting point of our stability analysis is the linear stability problem. The linear dynamics is governed by (4.2) with $\delta = 0$. For the sake of completeness we summarize now the linear problem ignoring $O(r^{2/3})$ effects.

$$\left[\frac{\partial}{\partial t} + (1+\tfrac{1}{2}\varepsilon+U_1)\frac{\partial}{\partial x}+V_1\frac{\partial}{\partial y}\right]q_1 + \frac{\partial\varphi_1}{\partial x}\frac{\partial\pi_1}{\partial y} - \frac{\partial\varphi_1}{\partial y}\frac{\partial\pi_1}{\partial x} = 0$$

$$\left[\frac{\partial}{\partial t} + (1-\tfrac{1}{2}\varepsilon-U_2)\frac{\partial}{\partial x}+V_2\frac{\partial}{\partial y}\right]q_2 + \frac{\partial\varphi_2}{\partial x}\frac{\partial\pi_2}{\partial y} - \frac{\partial\varphi_2}{\partial y}\frac{\partial\pi_2}{\partial x} = 0$$

$$q_n = \nabla^2\varphi_n + (-1)^n\beta(\varphi_1-\varphi_2) \qquad (5.1)$$

$$\varphi_n = 0 \quad \text{on } y = 0,1$$

$$t = 0 \quad \varphi_n = \varphi_n(x,y)$$

$$\nabla\pi_1 = [-(m^2\pi^2+\beta)V_1+\beta V_2]\hat{\imath} + [\beta(1+\varepsilon)+(m^2\pi^2+\beta)U_1-\beta U_2]\hat{\jmath}$$

$$\nabla\pi_2 = [-(m^2\pi^2+\beta)V_2+\beta V_1]\hat{\imath} + [\beta(1-\varepsilon)+(m^2\pi^2+\beta)U_2-\beta U_1]\hat{\jmath} \qquad (5.2)$$

where $U_n$ and $V_n$ are derived in the usual way from the x-dependent part of (3.12).

Non-parallel effects enter the linear stability problem through the meridional velocity $V_n$ and the zonal component of the potential vorticity gradient $\partial\pi_n/\partial x$ which are of $O(r^{1/3})$ and through the dependence on the zonal direction of the $O(1)$ zonal velocity.

In conventional stability studies the basic flow is assumed parallel and in the zonal direction. This amounts

to ignoring all $O(r^{1/3})$ effects in (5.1). The mathematical advantage of this assumption is that the problem becomes separable in x and normal mode solutions of the form $f_n(y)\exp i(kx-\omega t)$ can be sought. The faith in this approach is based on the notion that non-parallel effects can be ignored and that the physically realizable solution is the one with maximum temporal growth rate. Merkine and Balgovind, (1983) who studied numerically the barotropic analogue of (5.1), demonstrated the inapplicability of the parallel theory. They showed the strong controlling influence of the weak non-parallel effects on the evolution of the instability. In fact all the quantitative predictions of the parallel theory were proved to be incorrect. The only virtue of the parallel theory was in identifying the necessary conditions for instability. We adopt the same approach here, and in the next section we discuss the necessary conditions for instability derived from the parallel theory.

Another important point concerns critical layers. The linear parallel analysis leads to an eigenvalue problem determining the eigenvalue and eigenfunction corresponding to a given wavenumber k. In most circumstances the solutions of the inviscid parallel problem have a singular behavior at the critical layer where the phase speed of the instability is equal to the basic zonal velocity. The singularity is unacceptable physically and in the past viscous or non-linear effects were used to remove it. Merkine (1983) demonstrated that the singularity can be removed through the inclusion of non-parallel effects. He also showed that for the range of parameters typical of quasi-geostrophic flows the non-parallel critical layer is more important than its viscous or non-linear analogues.

## 6. The necessary conditions for instability of the linear parallel problem.

When the weak non-parallel effects are ignored in (4.1) and normal mode solutions of the type discussed earlier are assumed two necessary conditions for instability can be derived (Pedlosky, 1979). In our notation the two conditions are:

$$\frac{\partial \pi_n}{\partial y} \text{ must be somewhere positive and somewhere negative in the range of } y. \tag{6.1}$$

$$\left(1 + \frac{1}{2}\,\varepsilon(-1)^{n+1} + U_n\right)\frac{\partial \pi_n}{\partial y} \text{ must be somewhere positive in the range of } y. \tag{6.2}$$

If none of $\partial\pi_n/\partial y$ vanishes then they must be of opposite signs. This is the situation for baroclinic instability with no horizontal shear in a two layer model. On the other hand, if $\partial\pi_1/\partial y = \partial\pi_2/\partial y$ as in the barotropic problem then both must vanish at some point in order to satisfy the first condition for instability. In the general case the two possibilities can occur simultaneously implying equal importance of the baroclinic and barotropic processes. This is the situation typical of our model. Condition (6.2) has to be satisfied at least at one point in one of the layers.

We proceed now to apply conditions (6.1) - (6.2) to our particular basic state. From (3.11) - (3.12), (4.5) and (5.2) it follows that

$$\begin{aligned}\frac{\partial \pi_1}{\partial y} &= \beta(1+\varepsilon) + (m^2\pi^2+\beta)U_1 - \beta U_2 \\ \frac{\partial \pi_2}{\partial y} &= \beta(1-\varepsilon) + (m^2\pi^2+\beta)U_2 - \beta U_1\end{aligned} \tag{6.5}$$

where

$$U_1 = \frac{\Delta}{r^{2/3}} \frac{k_1}{6m^3\pi^3} \left\{\mu[\beta(1+\varepsilon)+m^2\pi^2(1-\varepsilon/2)]-\nu[\beta-m^2\pi^2(1-\varepsilon/2)]\right\} R(x)\cos m\pi y$$

$$U_2 = \frac{\Delta}{r^{2/3}} \frac{k_1}{6m^3\pi^3} \left\{\mu[\beta(1-\varepsilon)+m^2\pi^2(1+\varepsilon/2)]+\nu[\beta-m^2\pi^2(1+\varepsilon/2)]\right\} R(x)\cos m\pi y \quad . \tag{6.4}$$

$R(x)$, $k_1$ and $\beta$ are defined in (3.10). (6.3) and (6.4) reveal the richness of the stability problem. The potential vorticity gradients depend on $\varepsilon$ and m which fix $\beta$ and $k_1$ and on $\Delta/r^{2/3}$, $\mu$ and $\nu$ in addition to the parametric dependence on x. Our earlier choice of m = 2 does little to simplify the analysis. It seems reasonable to classify the first necessary condition for instability according to the sources which can be barotropic, baroclinic or of mixed type.

6.1 Barotropic sources, $\nu = 0$.

From (6.3) and (6.4) it follows, after some tedious manipulations, that

$$\frac{\partial\pi_1}{\partial y} = A_1(1+\tfrac{1}{2}\varepsilon+U_1) \;\; ; \;\; A_1 = m^2\pi^2 \; \frac{2\varepsilon(\beta/m^2\pi^2)^2+(\beta/m^2\pi^2)+1-\frac{1}{2}\varepsilon}{(1+\varepsilon)(\beta/m^2\pi^2)+1-\frac{1}{2}\varepsilon}$$

$$\frac{\partial\pi_2}{\partial y} = A_2(1-\tfrac{1}{2}\varepsilon+U_2) \;\; ; \;\; A_2 = m^2\pi^2 \; \frac{-2\varepsilon(\beta/m^2\pi^2)^2+(\beta/m^2\pi^2)+1+\frac{1}{2}\varepsilon}{(1-\varepsilon)(\beta/m^2\pi^2)+1+\frac{1}{2}\varepsilon} . \tag{6.5}$$

These expressions demonstrate the proportionality of the potential velocity gradients in each layer to the corresponding basic state's total velocity. The shear parameter $\varepsilon$ is restricted to the range $0 \le \varepsilon \le 2$ for which $A_1 > 0$. $A_2$ is positive for $\varepsilon < 1$ and negative for $\varepsilon > 1$. It follows that when the sources are sufficiently weak such that the total zonal velocity in each layer is of one sign the first necessary condition for instability is satisfied

when $\varepsilon > 1$. This is the same condition required for triggering inviscid baroclinic instability in a zonally uniform flow (see discussion following (2.13)). We expect, therefore, that in the case of weak sources in the sense discussed above, the necessary energy for the developing instability must come from the available potential energy of the zonally uniform part of the basic state.

When $\varepsilon < 1$, $A_2$ is positive and the first condition for instability can be satisfied only if the sources are sufficiently strong such that the zonal velocity reverses its direction somewhere within the range of y. When non-parallel effects are considered, this flow reversal is equivalent to the appearance of closed circulations (Merkine and Belgovind, 1983). It can be shown that closed circulations appear first in the lower layer. The maximum value of R(x) in the expressions for $U_n$ is $\frac{1}{2}3^{\frac{1}{2}}\exp(-\pi/6 \cdot 3^{\frac{1}{2}}) \simeq 0.64$ and it is attained at $\frac{1}{2}3^{\frac{1}{2}}k_1 r^{1/3} x_2^n = \pi/6$. It follows that the first necessary condition for instability is satisfied whenever

$$1 - \frac{1}{2}\varepsilon - \frac{\Delta}{r^{2/3}} \frac{0.64\ k_1}{6m\pi} \left[\frac{\beta}{m^2\pi^2}(1-\varepsilon) + 1 + \frac{1}{2}\varepsilon\right] < 0$$

which sets the strength of the barotropic potential vorticity sources for given $\varepsilon$ and m. The last expression indicates that stronger shear requires weaker sources for satisfying the necessary condition for instability.

In the strong-shear weak-source case discussed earlier the lower layer potential vorticity gradient becomes negative everywhere when $\varepsilon > 1$. In the weak-shear strong-source case just discussed the potential vorticity gradient becomes negative in isolated regions which appear first in the lower layer. This situation for which the sign reversal occurs within a given layer is typical of two-layer barotropic processes. We expect that in such circumstances the

**energy source for the instability will be the available kinetic energy of the basic state.** When $\varepsilon > 1$ and the basic state possesses closed circulations baroclinic and barotropic processes must be of comparable importance and the developing instability should be of a mixed type. This is a rather complex situation the stability properties of which are not very well understood.

6.2 Baroclinic sources, $\mu = 0$.

When the sources are baroclinic the potential vorticity gradient can be written as

$$\frac{\partial \pi_1}{\partial y} = \beta(1+\varepsilon) - B_1 U_1 \; ; \; B_1 = m^2\pi^2 \, \frac{2(\beta/m^2\pi^2)^2 - 3(\beta/m^2\pi^2) + 1 - \frac{1}{2}\varepsilon}{(\beta/m^2\pi^2) - 1 + \frac{1}{2}\varepsilon}$$

$$\frac{\partial \pi_2}{\partial y} = \beta(1-\varepsilon) - B_2 U_2 \; ; \; B_2 = m^2\pi^2 \, \frac{2(\beta/m^2\pi^2)^2 - 3(\beta/m^2\pi^2) + 1 + \frac{1}{2}\varepsilon}{(\beta/m^2\pi^2) - 1 - \frac{1}{2}\varepsilon}. \tag{6.6}$$

We observe that unlike the barotropic case the potential vorticity gradients are not proportional to the basic zonal velocity. When $\varepsilon = 0$, $U_n \equiv 0$ regardless of the ratio $\Delta/r^{2/3}$. This result is not surprising. $U_n$ is the resonating part of the basic state. The zero group velocity resonance is associated with the barotropic branch of the dispersion relation which in the absence of shear cannot trigger baroclinic response. The response triggered by the baroclinic sources when $\varepsilon = 0$ is $O(\Delta)$. This is a very weak response and it is not relevant for our stability study since it modifies the zonally uniform part of the basic state by an $O(\Delta)$ only. Such small contributions are not included in our description of the basic state. Thus when $\varepsilon = 0$, $\partial\pi_n/\partial y = \beta$ and the basic state is stable. In the absence of sources $\varepsilon > 1$ is necessary for instability. This is identical to the corresponding barotropic case and the instability is purely baroclinic.

In the case of barotropic sources the criterion $\varepsilon > 1$ holds as long as $U_n$ is sufficiently weak such that no closed circulation is induced by the sources. The situation is different in the present case. A weak $U_n$ can alter significantly the stability criterion and, in fact, we expect it to destabilize the flow. This is illustrated most easily for the case of $\varepsilon = 1$ for which $\partial\pi_2/\partial y = 2m^2\pi^2U_2$ $(B_2(\varepsilon = 1) = -2m^2\pi^2)$. We see that no matter how small $U_2$ is, its meridional structure which is proportional to $\cos(m\pi y)$ ensures the vanishing of the lower layer potential vorticity gradient at at least one point. In this case the baroclinic energy criterion is altered but the energy source of the instability is likely to be the available kinetic energy of the basic state. However when $\varepsilon > 1$ and $U_2$ is very small the instability should be baroclinic in nature. In the combined case of $\varepsilon > 1$ and strong $U_n$ the instability is expected to be of mixed type.

The one to one correspondence between the closed circulations of the basic state and the closed contours of the potential vorticity gradients that we observed in Section 6 does not exist in the case of baroclinic sources. However, in both cases closed circulations appear first in the lower layer where the uniform part of the zonal flow is weaker. At the end of the report we provide a few figures which illustrate typical flow fields and the associated potential vorticity gradients.

The above discussion has demonstrated the complexity of the stability problem and there is no point for analyzing now the stability criteria for the general case when the sources are of a mixed type. Further advancement in our understanding of the energetics of the instability can only be achieved by actually solving the stability problem.

In this section and in the previous one all the information was extracted by applying the first necessary condition for instability to our particular basic state. The second necessary condition for instability can always be trivially satisfied and hence it is not very informative.

## 7. The numerical model and a few preliminary results.

The stability problem formulated in Section 4 must be integrated numerically. We describe briefly now the numerical scheme and present a few preliminary results pertinent to the linearized version of the stability problem. The numerical code was developed by Y. Bar-Sever who is a graduate student participating in the research described in this report. The numerical method is an extention to the two layer model of the numerical scheme described by Kalnay and Merkine (1981).

The numerical scheme developed for the initial boundary value problem stated in Section 4 conserves potential enstrophy. A spatial staggered grid is used where the prognostic variable $q_n$ (the potential vorticity of the perturbation) is defined at the center of the grid and $\varphi_n$ (the streamfunction of the perturbation) is defined at the corners of the grid. This allows a direct implementation of the meridional boundary conditions at $y = 0,1$ which require the vanishing of the perturbation streamfunction. The zonal extent of the numerical channel is $-20 \leq x \leq 20$ and the sources are placed at $x = 0$. The time difference scheme is the N-cycle (Lorenz, 1971) with $N = 4$. The potential vorticity of the perturbation is updated at the numerical zonal boundaries of the channel by extrapolating it linearly outward from the interior, i.e. in the direction of the group velocity of the waves which are excited at the central portion of the channel. In order to minimize spurious reflexion of waves from the zonal boundaries of the domain of integration we used the sponge layer method (Kalnay and Merkine, 1981) whereby the time scale for dissipative effects is decreased gradually toward the zonal boundaries. The perturbation steamfunction at the zonal boundaries is determined by imposing the condition $\partial^2\varphi_n/\partial^2x^2 = 0$ (which amounts to linear extrapolation) in the definition of the updated potential vorticity evaluated at the zonal boundaries. This leads to a pair of ordinary

differential equations which determine $\varphi_n$ at the zonal boundaries. The updated interior perturbation streamfunction is obtained from the updated interior potential vorticity using the Fishpak Algorithm 541 for solving elliptic systems subject to the boundary conditions imposed on $\varphi_n$ as stated above. This algorithm was developed at NCAR by P. N. Swarztrauber (1979).

The sponge layer device and the procedure for evaluating $q_n$ and $\varphi_n$ at the zonal boundaries of the domain of integration are successful only if the central portion of the flow domain is little affected by end effects. The results of Kalnay and Merkine (1983) demonstrate that this is indeed the case. However, each numerical integration must be scrutinized separately.

The consistency of the numerical scheme was checked by performing several test runs. For example, numerical integrations were performed for the case of super-critical shear, no sources and periodic zonal boundary conditions. This is the classical problem of two layer baroclinic instability. The initial perturbation was a localized pulse. Baroclinic instability developed. The growth rate and phase speed of the most rapidly growing mode agreed well with the predictions of the dispersion relation (2.11). In other test runs the results of Merkine and Balgovind (1983) were reproduced. In all runs the spatial grid size was uniform and equal to 0.1 which provides a reasonable resolution for the phenomena under investigation. This amounts to 4000 grid points for each layer. For the range of parameters considered so far numerical stability was ensured by choosing 0.05 as the time increment for every 4 cycles of the Lorenz scheme. In many cases it is necessary to integrate to about 20 time units. Using the Technion's IBM 3081D computer a typical CPU time for such a run is about 15 minutes. It was impractical to check convergence by decreasing the grid size. We rely on the conclusion of Merkine and Balgovind (1983) which states that only quantitative but no qualitative changes were observed when a finer spatial grid was used.

We proceed now to describe some preliminary results pertinent to the linear phase of the evolution of the instability. In particular, we consider the influence of $\varepsilon$, which is the shear parameter of the basic state at infinity, on the evolution of the instability. The results presented here correspond to positive barotropic sources i.e. $\mu = 1$ and $\nu = 0$ with $\Delta = 0.1$ and $r = 0.001$. Figures 2 and 3 depict the streamfunctions of the basic state corresponding to $\varepsilon = 0.5$ and $\varepsilon = 1.5$, respectively. The basic state describes a blocking configuration. The sources induce a closed circulation at the center of the channel. As a consequence, the on-coming westerly flow is split into two intense jets flowing along the meridional boundaries of the domain. For the parameters chosen the closed circulation appears in both layers but it is larger in the lower layer which is characterized by a weaker flow at infinity. The two basic states depicted in Figures 2 and 3 are broadly similar but with one difference. In the absence of sources the basic state is baroclinically stable for $\varepsilon = 0.5$ and unstable for $\varepsilon = 1.5$. Based on the discussion of Section 6.1 closed circulation is necessary for instability when $\varepsilon = 0.5$ but not when $\varepsilon = 1.5$. Thus the instability is expected to be basically barotropic when $\varepsilon = 0.5$ and of a mixed type where $\varepsilon = 1.5$.

The results of the numerical integration reveal the existence of an unstable localized wave packet characterized by a single complex eigenfrequency, $\omega$. For $\varepsilon = 0.5$ $\omega = 0.84+1.88i$ and for $\varepsilon = 1.5$ $\omega = 1.00+2.54i$. It follows that the increase in the baroclinicity of the basic state enhances the instability. However, without calculating the energy transfer mechanisms we cannot be certain whether the enhanced growth rate is due to a release of the available potential energy of the basic state or that the increase in $\varepsilon$ affects a greater release of the available kinetic energy of the basic state. The energy calculations will be reported elsewhere as a part of a detailed examination of the stability problem. Figures 4-7 suggest, however, that

the instability is essentially barotropic. In Figures 4 and 5 we see the streamfunction of the instability for $\varepsilon = 0.5$ and it is very similar to the barotropic instability of Merkine and Balgovind (1983). The cells are bowed to the east corresponding to a Reynolds stress distribution which is positively correlated with the horizontal shear of the basic state. The absence of phase changes in the vertical (Figure 5) indicates that the instability is barotropic. The effect of the vertical shear is more noticeable in Figures 6 and 7 which correspond to $\varepsilon = 1.5$. The upper layer streamfunction depicted in Figure 6a is similar to Figure 4a. It indicates release of the upper layer available kinetic energy of the basic state. The situation is different in the lower layer. The cells shown in Figure 6b are bowed in a way favorable for transfer of kinetic energy from the lower layer instability to the lower layer basic state. From Figure 7 it follows that the lower layer field lags slightly behind the upper layer field and this suggests that potential energy is transferred from the instability to the basic state. It must be emphasized that the above considerations are based on the parallel flow assumption and consequently must be supported by a more detailed study. However, if these conclusions hold then the increase of baroclinicity can occasionally enhance instability through barotropic processes.

One other important feature revealed by Figures 4-7 is the crisis region upstream of the maximum. This region is characterized by a rapid transition from long to short waves. As stated by Merkine and Balgovind (1983) the dynamics of this crisis region must necessarily be controlled by non-parallel effects. This concludes the brief discussion of some of our preliminary results.

## 8. Summary and prognosis

In the previous sections we have reported on our research efforts during the first year of support by the AFOSR. Briefly stated, we have developed an analytical three dimensional basic state whose zonal extent and baroclinic and barotropic characteristics can be varied at will. This basic state can be one prototype model for studying systematically the evolution of local quasi-geostrophic instabilities of a general type. Although the basic state is rather simple the structure is such that the evolution of the instability must be studied numerically. Consequently, we have developed a numerical model of sufficient flexibility for studying all the aspects of the instability. We have made a few preliminary runs and the results revealed the existence of a localized instability. During the second year of research (March 1, 1984 - February 28, 1985) we shall carry out a systematic study of all aspects of the linear phase of the instability. In particular, we shall examine the way the instability is affected by the strength of the sources and their structure as well as by the zonal length-scale of the basic state and the vertical shear at infinity. The third year of research (March 1, 1985 - February 28, 1986) will be devoted to the non-linear stability problem. The non-linear study deals with the equilibration of the instability and the distortion of the basic state. These two related processes are necessarily accompanied by wave radiation from the region of origin of the instability.

## References


Charney, J. G. 1954 The dynamics of long waves in a baroclinic westerly current, J. Meteor. 4, 135-162.

Charney, J. G. & Devore, J. G. 1979 Multiple flow equilibria in the atmosphere and blocking, J. Atmos. Sci. 36, 1205-1216.

Egger, J. 1978 Dynamics of blocking highs, J. Atmos. Sci. 35, 1788-1801.

Fredriksen, J. S. 1983 Disturbances and eddy fluxes in Northern Hemisphere flows; Instability of three-dimensional January and July flows, J. Atmos. Sci. 40, 835-855.

Greenspan, H. P. 1968 The Theory of Rotating Fluids, Cambridge University Press.

Holopainen, E. O. 1961 On the effect of friction in baroclinic waves, J. Atmos. Sci. 13, 363-367.

Illari, L. & Marshall, J. C. 1983 On the interpretation of eddy flows during a blocking episode, J. Atmos. Sci. 40, 2232-2242.

Kalnay, E. & Merkine, L. 1981 A simple mechanism for blocking, J. Atmos. Sci. 38, 2077-2091.

Lorenz, E. N. 1971 An N-cycle time-differencing scheme for stepwise numerical integration, Mon. Wea. Rev. 99, 644-648.

Manabe, S. and Terpstra, T. B. 1974 The effects of mountains on the general circulation of the atmosphere as identified by numerical experiments, J. Atmos. Sci. 31, 3-42.

Merkine, L. 1977 Convective and absolute instability of baroclinic eddies, Geophys. Astrophys. Fluid Dyn. 9 ,129-157.

Merkine, L. 1980 Linear and nonlinear resonance of Rossby waves, Geophys. Astrophys. Fluid Dyn. 15, 53-64.

Merkine, L. & Shafranek, M. 1980 The spatial and temporal evolution of localized unstable baroclinic disturbances, Geophys. Astrophys. Fluid Dyn. 16, 175-206.

Merkine, L. 1981 Resonance induced by Rossby lee waves, Geophys. Astrophys. Fluid Dyn. 18, 25-37.

Merkine, L. 1982 The stability of quasi-geostrophic fields induced by potential vorticity sources, J. Fluid Mech. 116, 315-342.

Merkine, L. 1983 A critical layer dominated by non-parallel effects in a rotating barotropic flow. Proc. R. Soc. Lond. A388, 293-310.

Merkine, L. & Balgovind, R. 1983 Barotropic instability of weakly non-parallel zonal flows. Geophys. Astrophys. Fluid Dyn. 25, 157-190.

Pedlosky, J. 1970 Finite amplitude baroclinic waves, J. Atmos. Sci. 37, 15-31.

Pedlosky, J. 1979 Geophysical Fluid Dynamics Springer-Verlag, New York

Petterssen, S. 1956 Weather Analysis and Forecasting, Vol. 1 McGraw Hill

Pierrehumbert, R.T. 1983 Local and global baroclinic instability of zonally varying flow. Preprint.

Rex, D.F. 1950a Blocking action in the middle troposphere and its effect upon regional climate. I. An aerological study of blocking action. Tellus 2, 196-301.

Rex, D. F. 1950b Blocking action in the middle troposphere and its effect upon regional climate. II. The climatology of blocking action, Tellus 2, 275-301.

Swarztrauber, P.N. 1974 Fishpak algorithm 541, ACM Transaction on Mathematical Software, 5, 352-364.

Tung, K. K. & R. S. Lindzen 1979 A theory of stationary long waves. Part I: A simple theory of blocking, Mon. Wea. Rev. 107, 714-734.

## Figure captions

Figure 1: A blocking configuration over the Atlantic.

Figure 2: The streamfunction of the basic state induced by a positive barotropic source, i.e. $\mu = 1$, $\nu = 0$. The shape of the source is given by $\delta(x)\sin 2\pi y$. In this example $\Delta = 0.1$, $r = 0.001$ and $\varepsilon = 0.5$. a) upper layer, b) lower layer.

Figure 3: The same as in Figure 2 but for $\varepsilon = 1.5$.

Figure 4: The streamfunction of the instability corresponding to the basic state depicted in Figure 2. a) upper layer, b) lower layer. The amplitude of the instability is arbitrary since the problem is homogeneous. The cells propagate eastward through a stationary envelope. The eigenfrequency is $\omega = 0.84+1.88i$. The maximum of both fields is rescaled to 100. The maximum of the lower layer field is 87% of the maximum of the upper layer field.

Figure 5: Cross sections at $y = 1/2$ of the fields shown in Figure 4. Full line corresponds to the upper layer and dashed line to the lower layer.

Figure 6: The streamfunction of the instability corresponding to the basic state depicted in Figure 3. a) upper layer, b) lower layer. The amplitude of the instability is arbitrary since the problem is homogeneous. The cells propagate eastward through a stationary envelope. The eigenfrequency is $\omega = 1.00+2.54i$. The maximum of both fields is rescaled to 100. The maximum of the lower layer field is 53% of the maximum of the upper layer field.

Figure 7: Cross sections at $y = 1/2$ of the fields shown in Figure 6. Full line corresponds to the upper layer and dashed line to the lower layer.

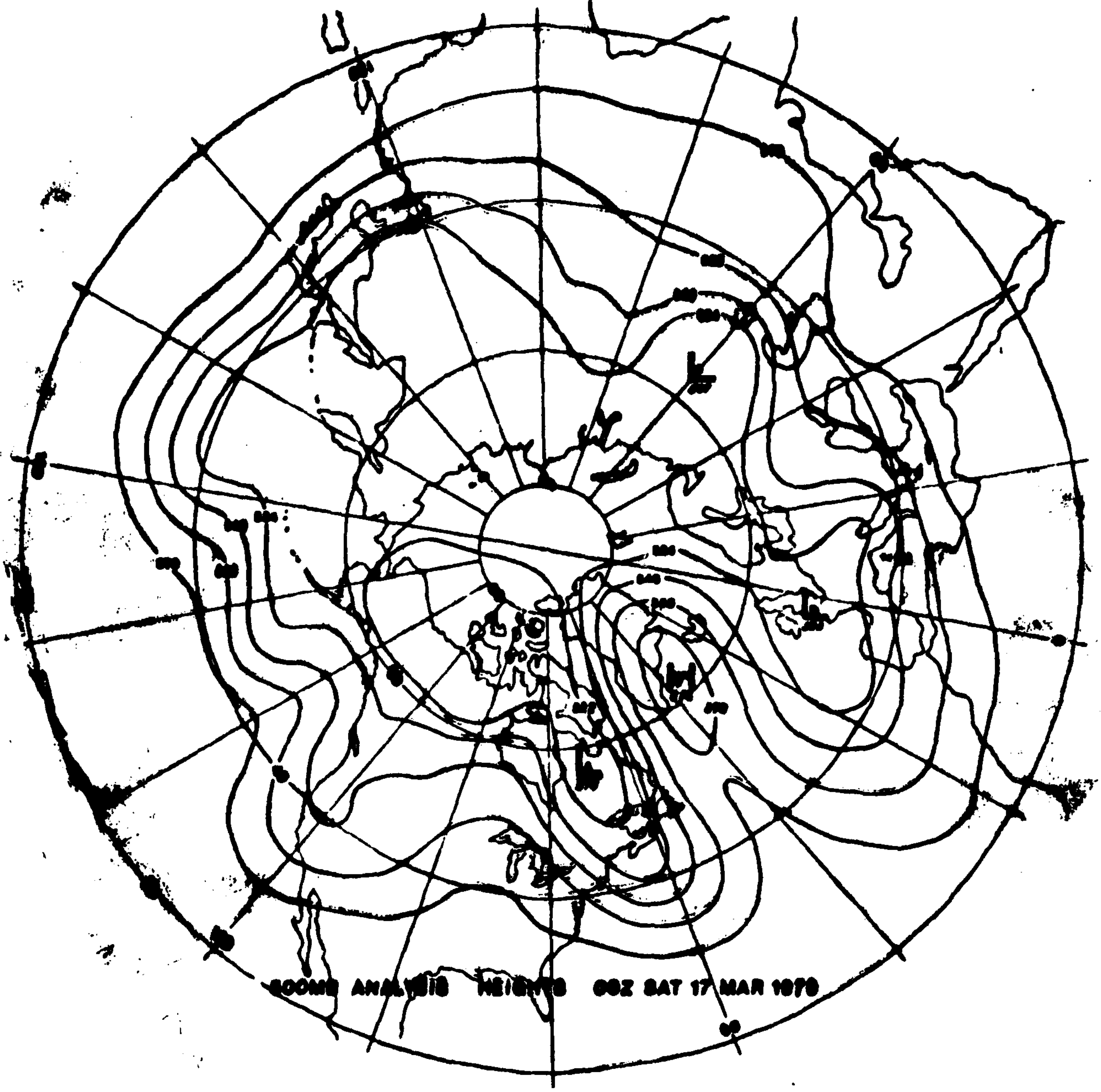
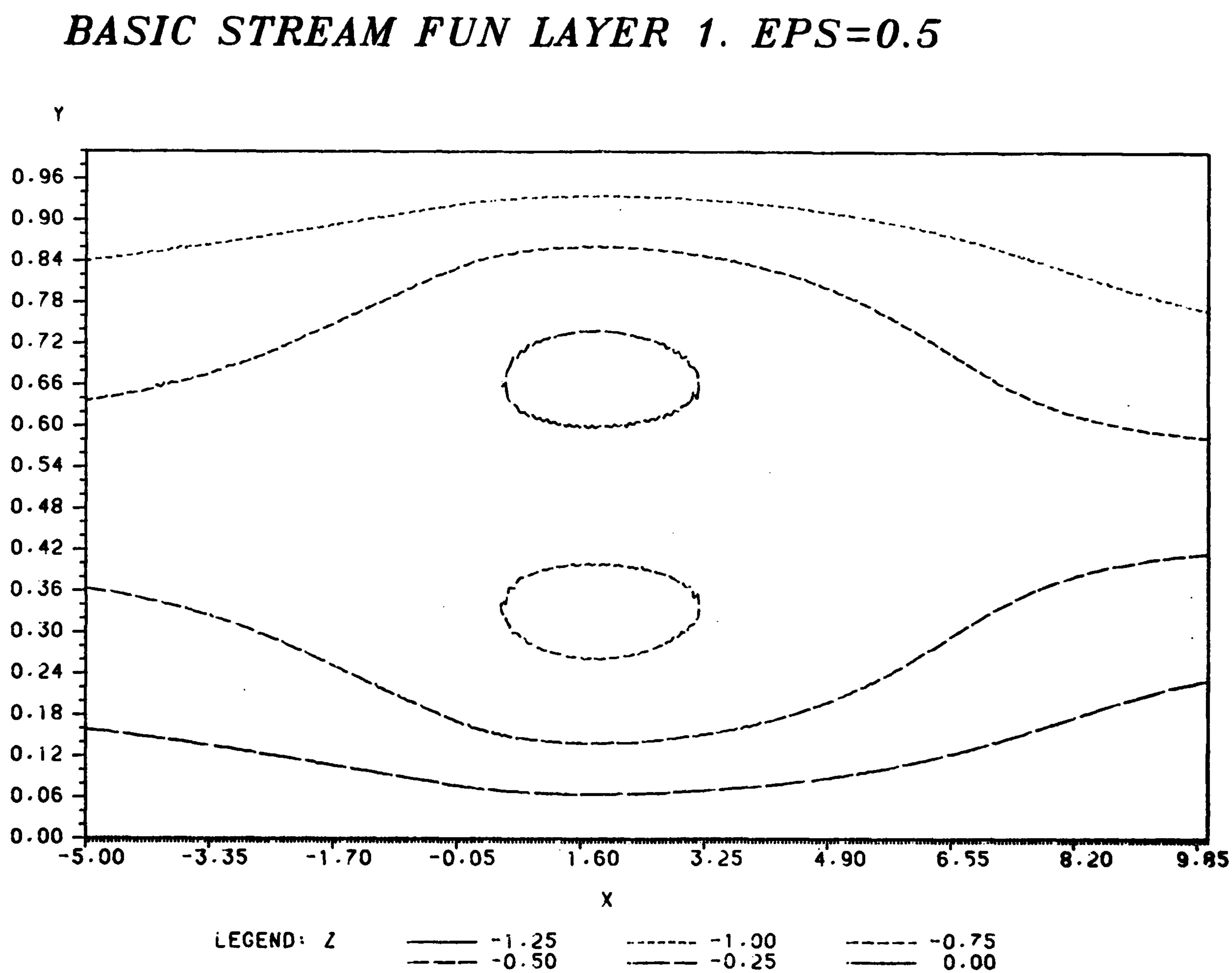



Fig. 1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0.48
0.42
0.36
0.30
0.24
0.18
0.12
0.06
0.00
-5.00
-3.35
-1.70
-0.05
1.60
3.25
4.90
6.55
8.20
9.85
X
LEGEND: Z
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-0.25
0.00

Fig. 2a

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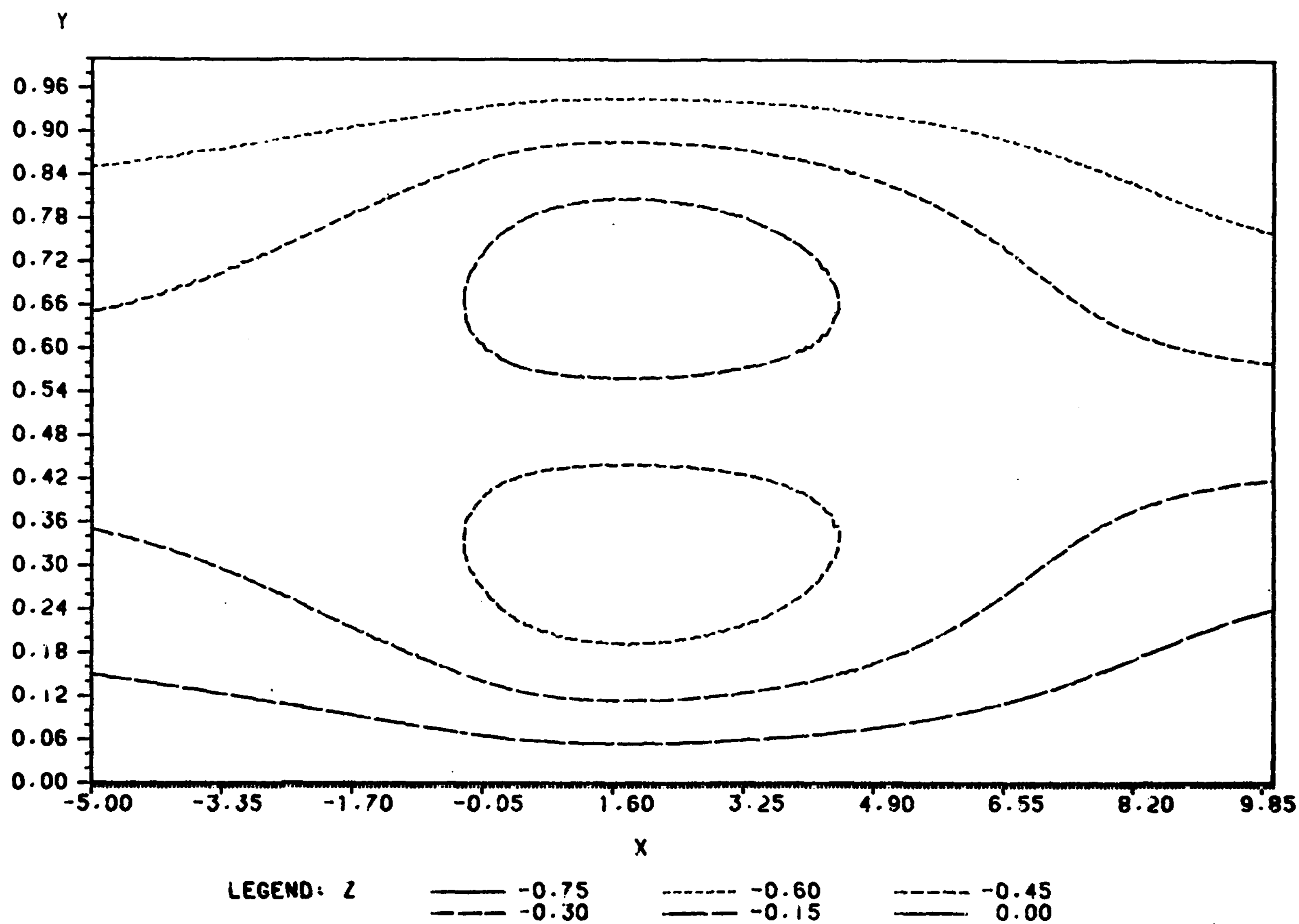
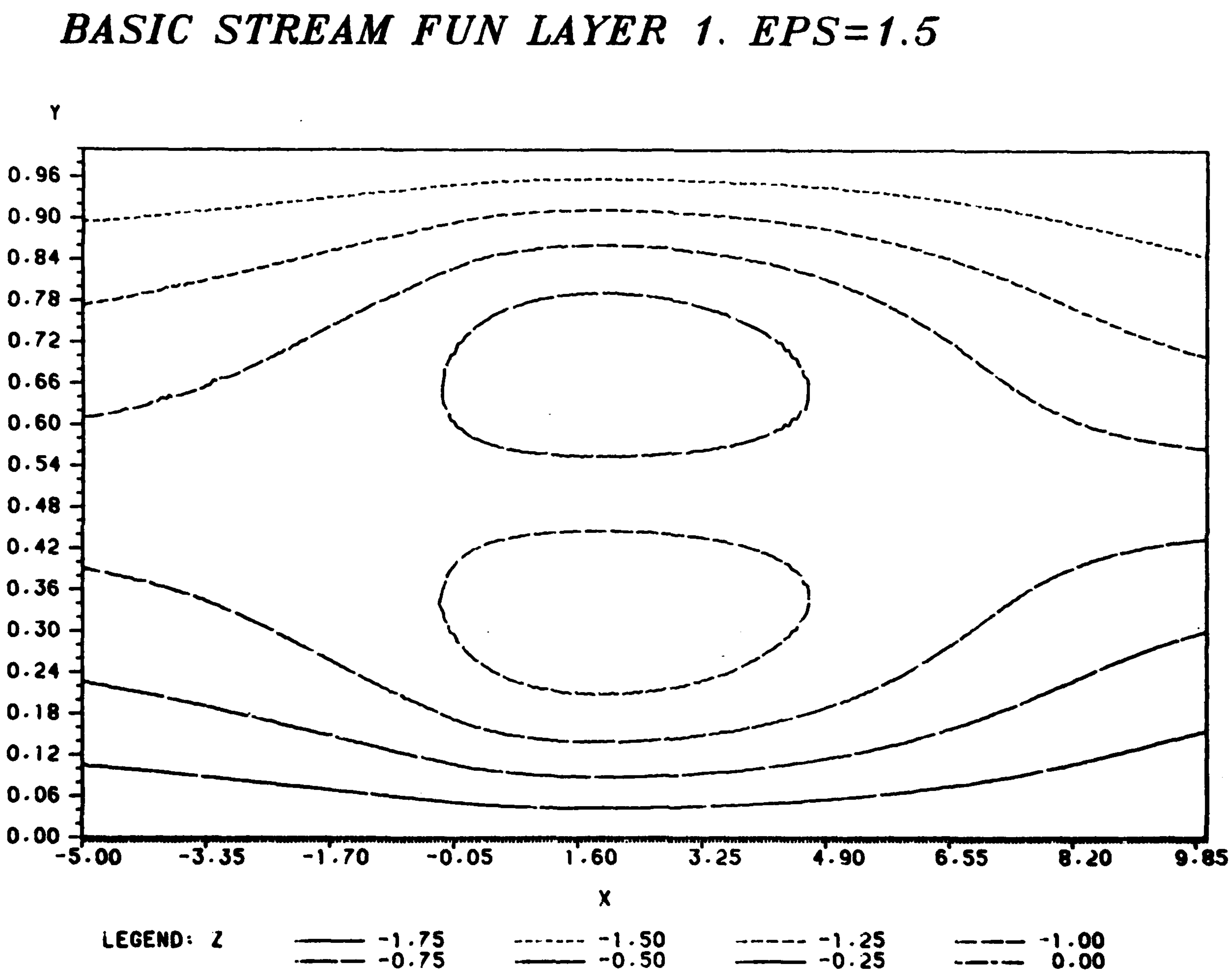



Fig. 2b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0.42
0.36
0.30
0.24
0.18
0.12
0.06
0.00
-5.00
-3.35
-1.70
-0.05
1.60
3.25
4.90
6.55
8.20
9.85
X
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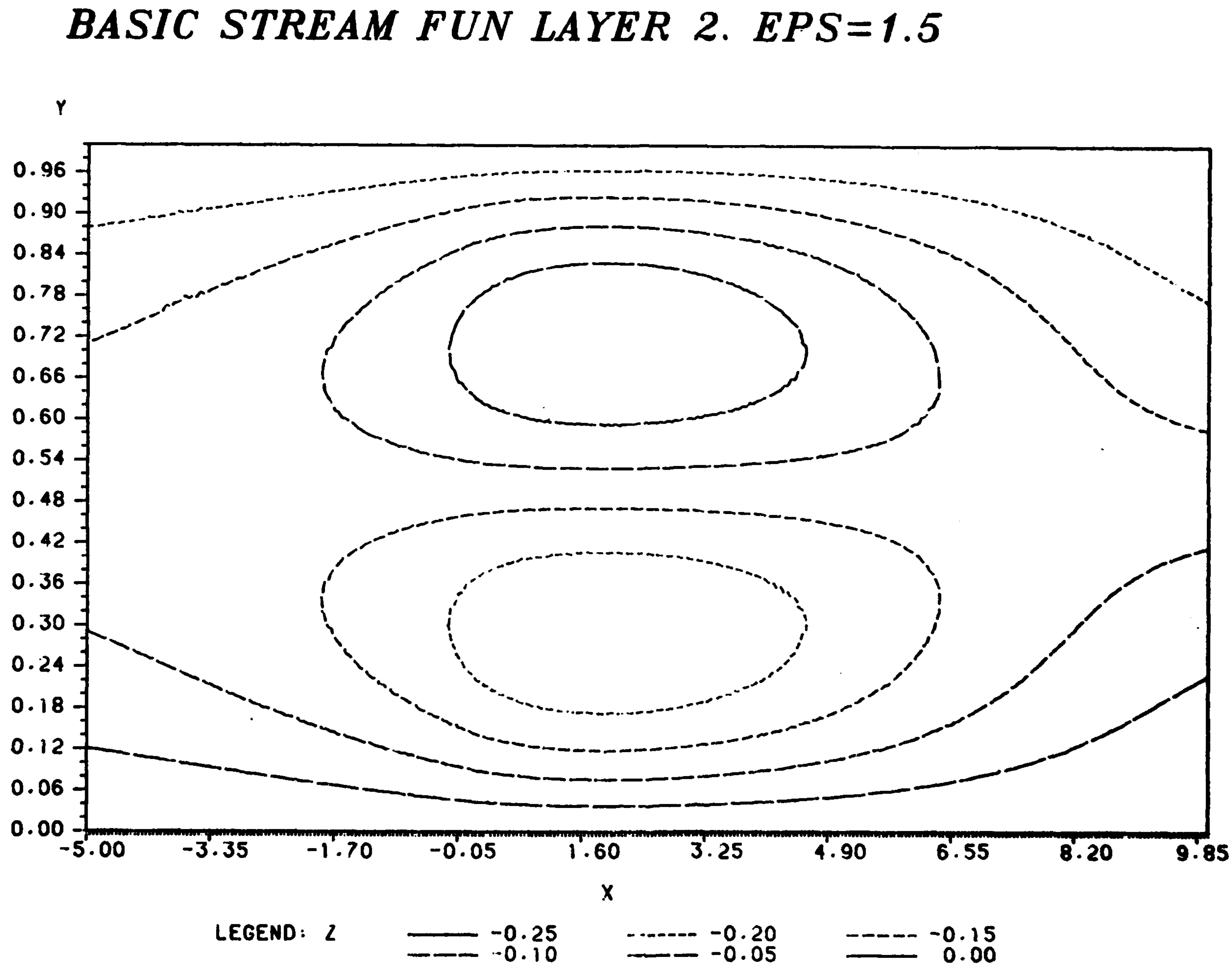
0.00

Fig. 3a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0.42
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0.30
0.24
0.18
0.12
0.06
0.00
-5.00
-3.35
-1.70
-0.05
1.60
3.25
4.90
6.55
8.20
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X
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-0.20
-0.15
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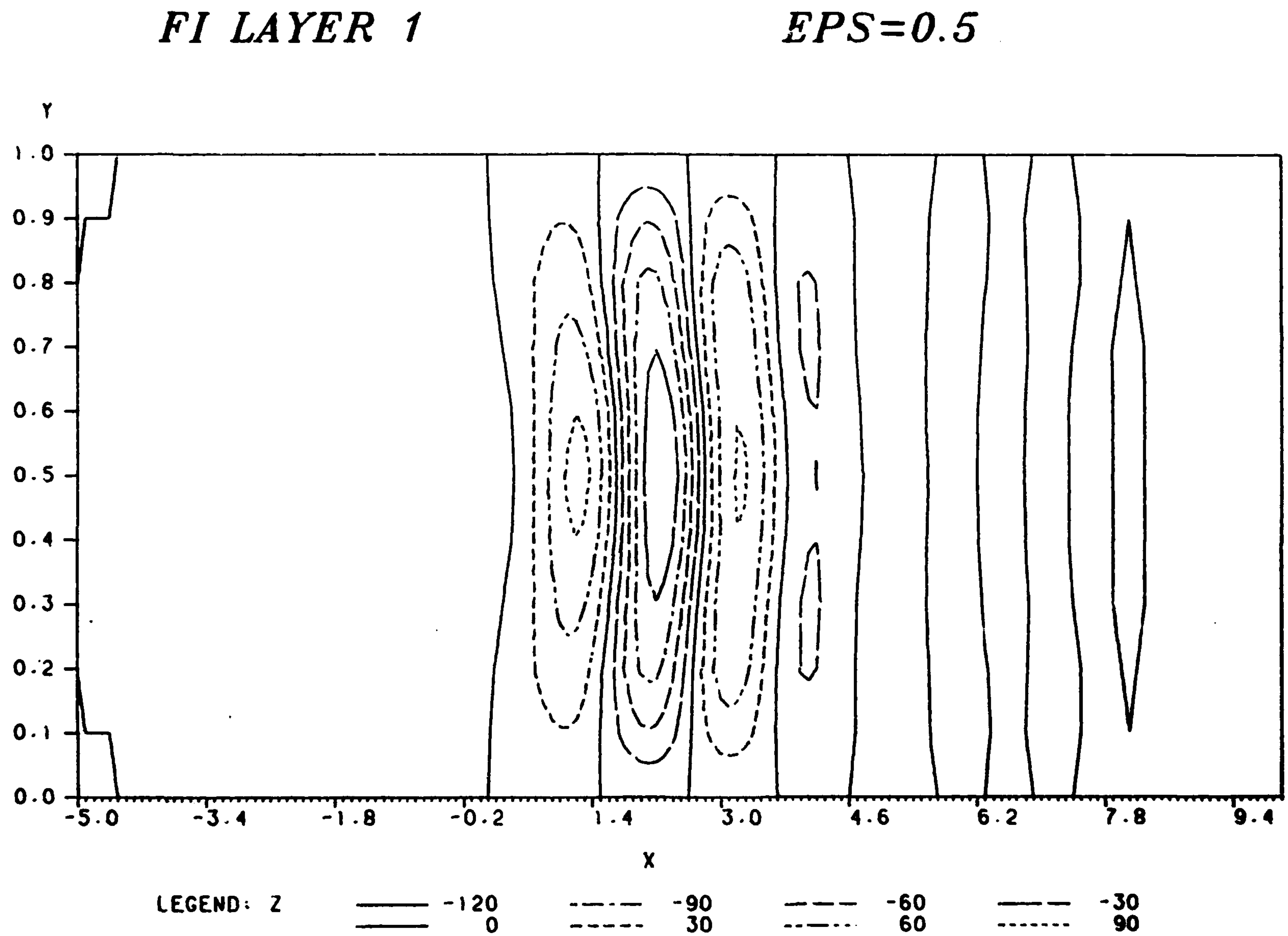
-0.05
0.00

Fig. 3b



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0.2
0.1
0.0
-5.0
-3.4
-1.8
-0.2
1.4
3.0
4.6
6.2
7.8
9.4
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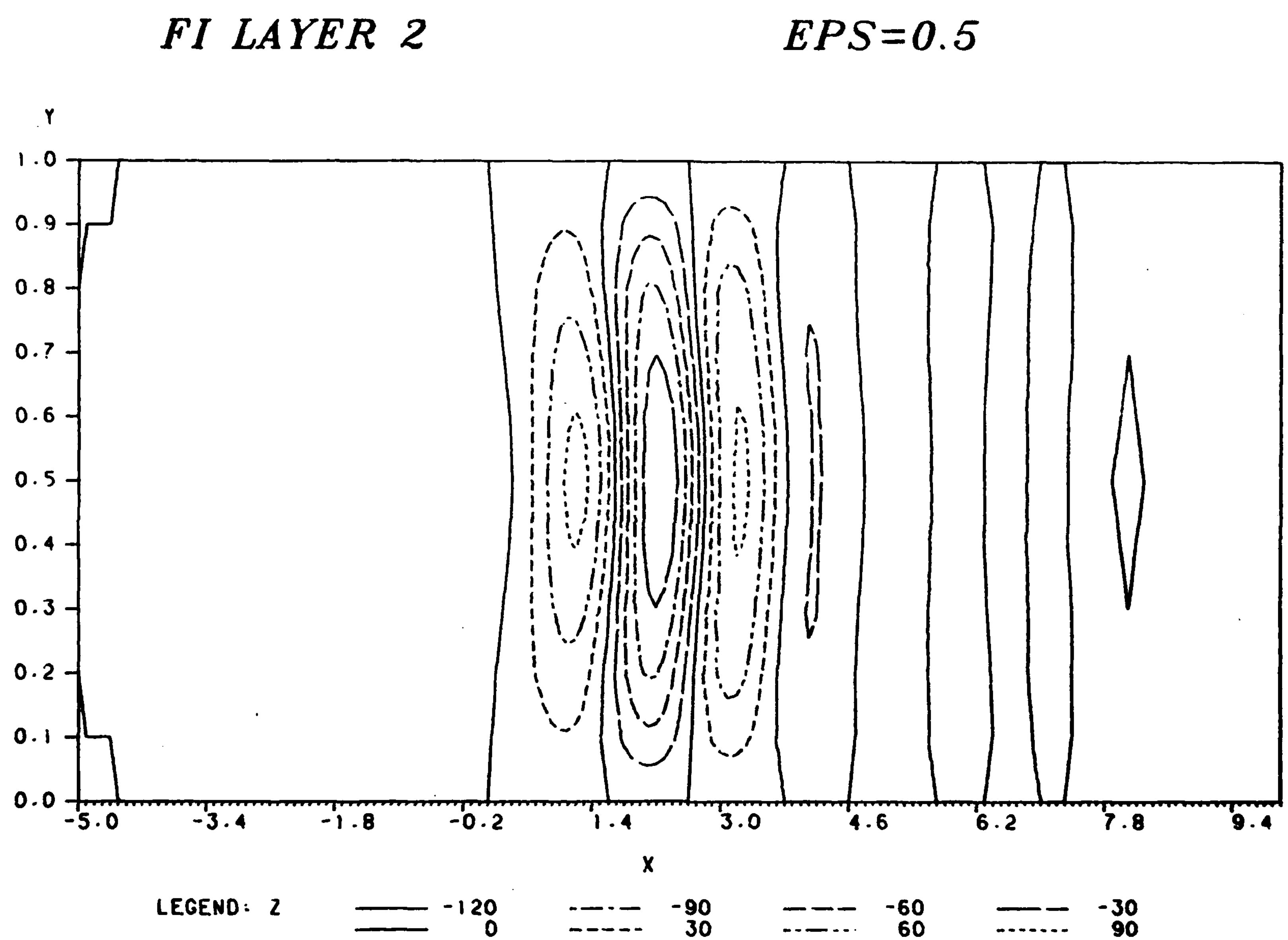
0
-90
30
-60
60
-30
90


Fig. 4a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-0.2
1.4
3.0
4.6
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7.8
9.4
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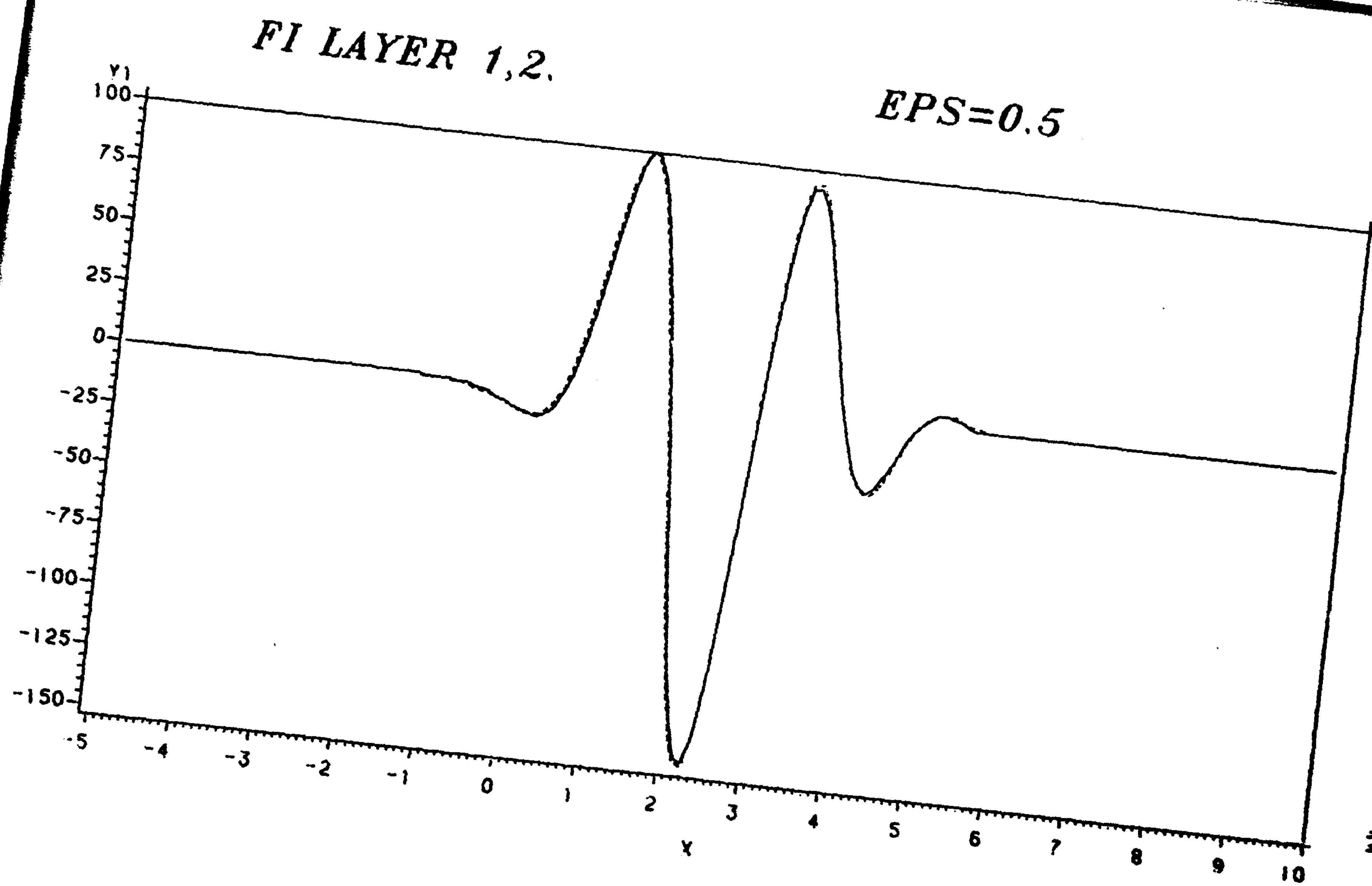
30
-60
60
-30
90

Fig. 4b



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-125
-150
-5
-4
-3
-2
-1
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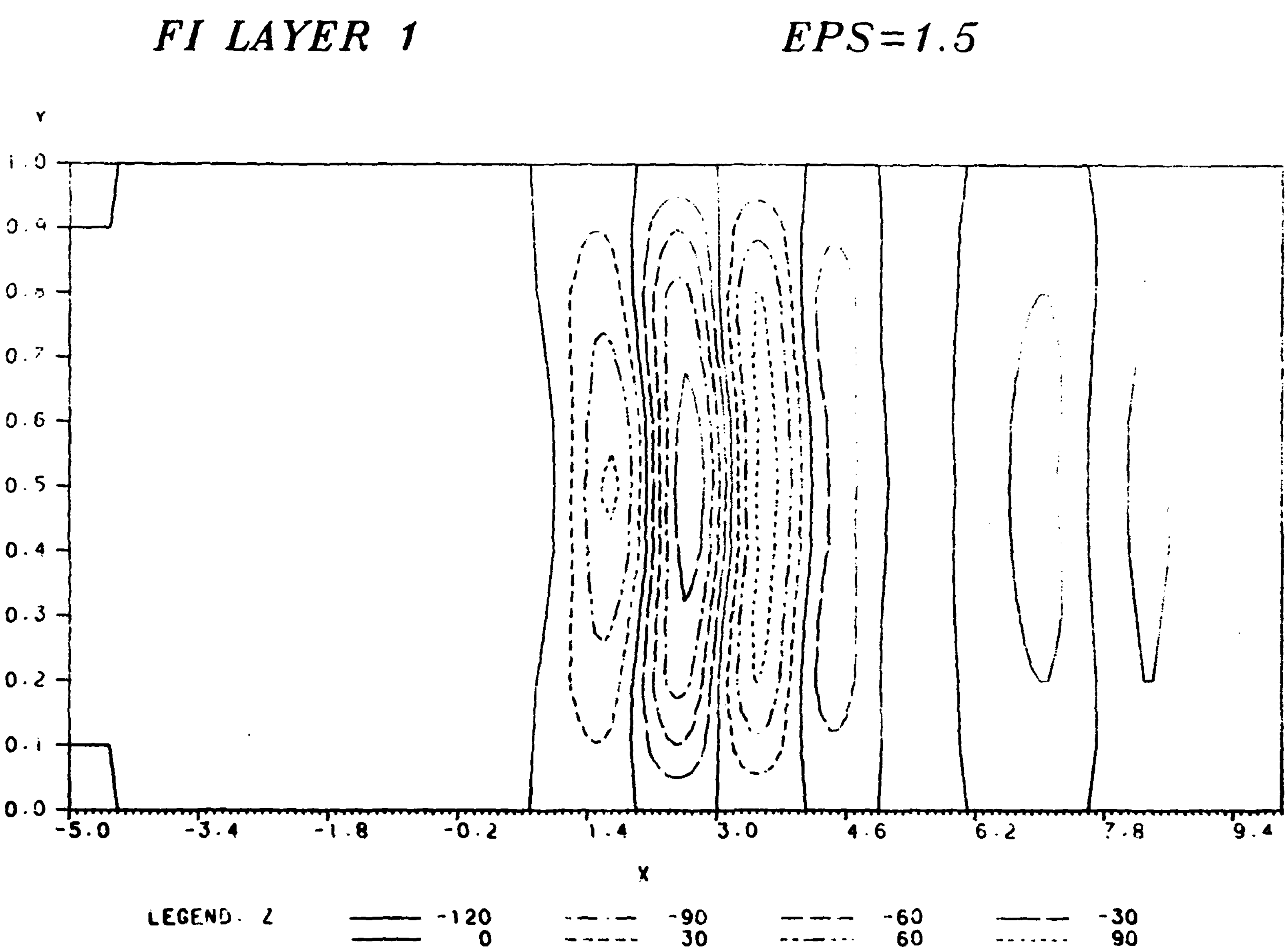
9
10
X


Fig. 5

44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0.1
0.0
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-0.2
1.4
3.0
4.6
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9.4
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LEGEND Z
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0
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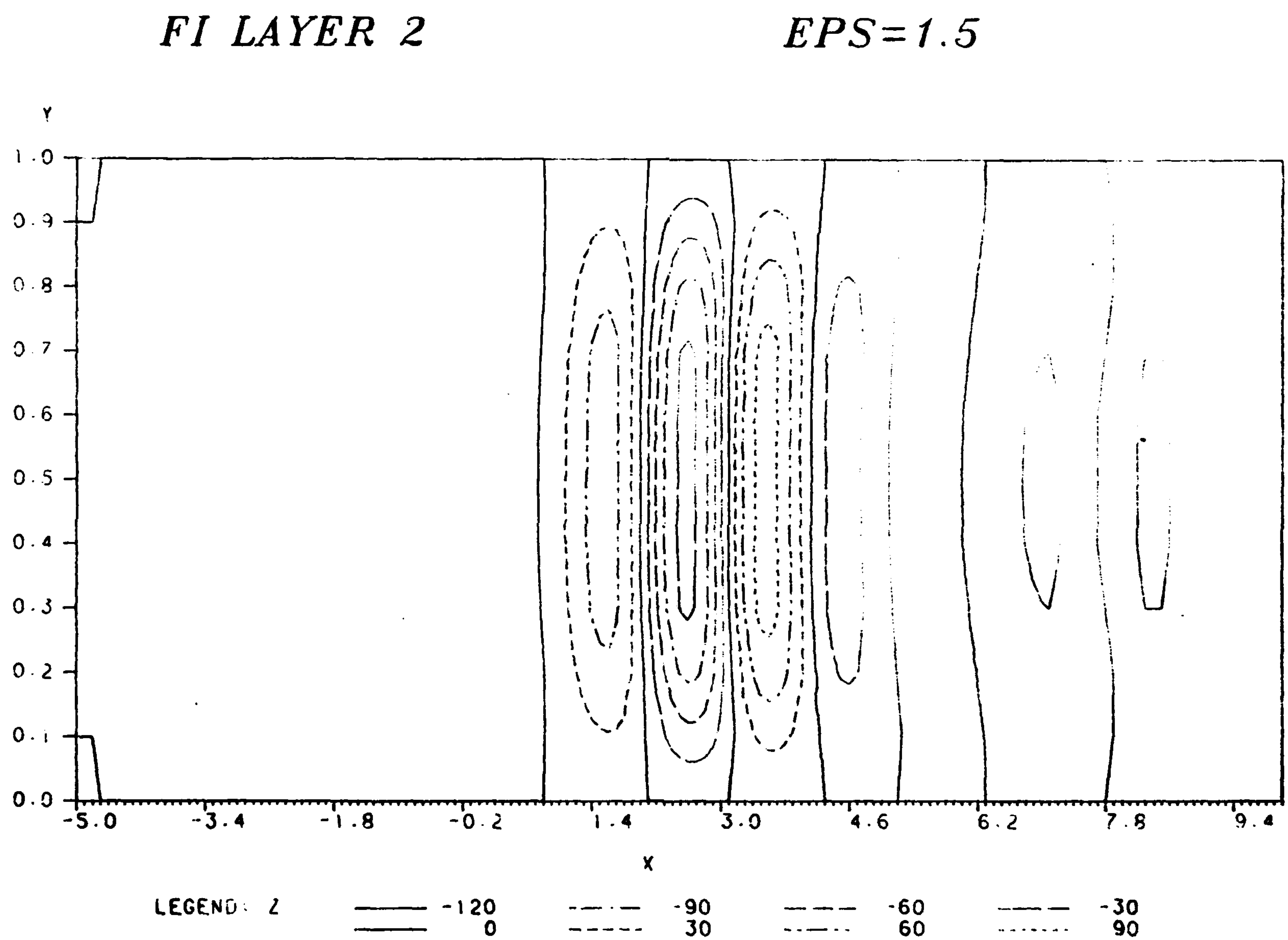
30
-60
60
-30
90

Fig. 6a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1.4
3.0
4.6
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9.4
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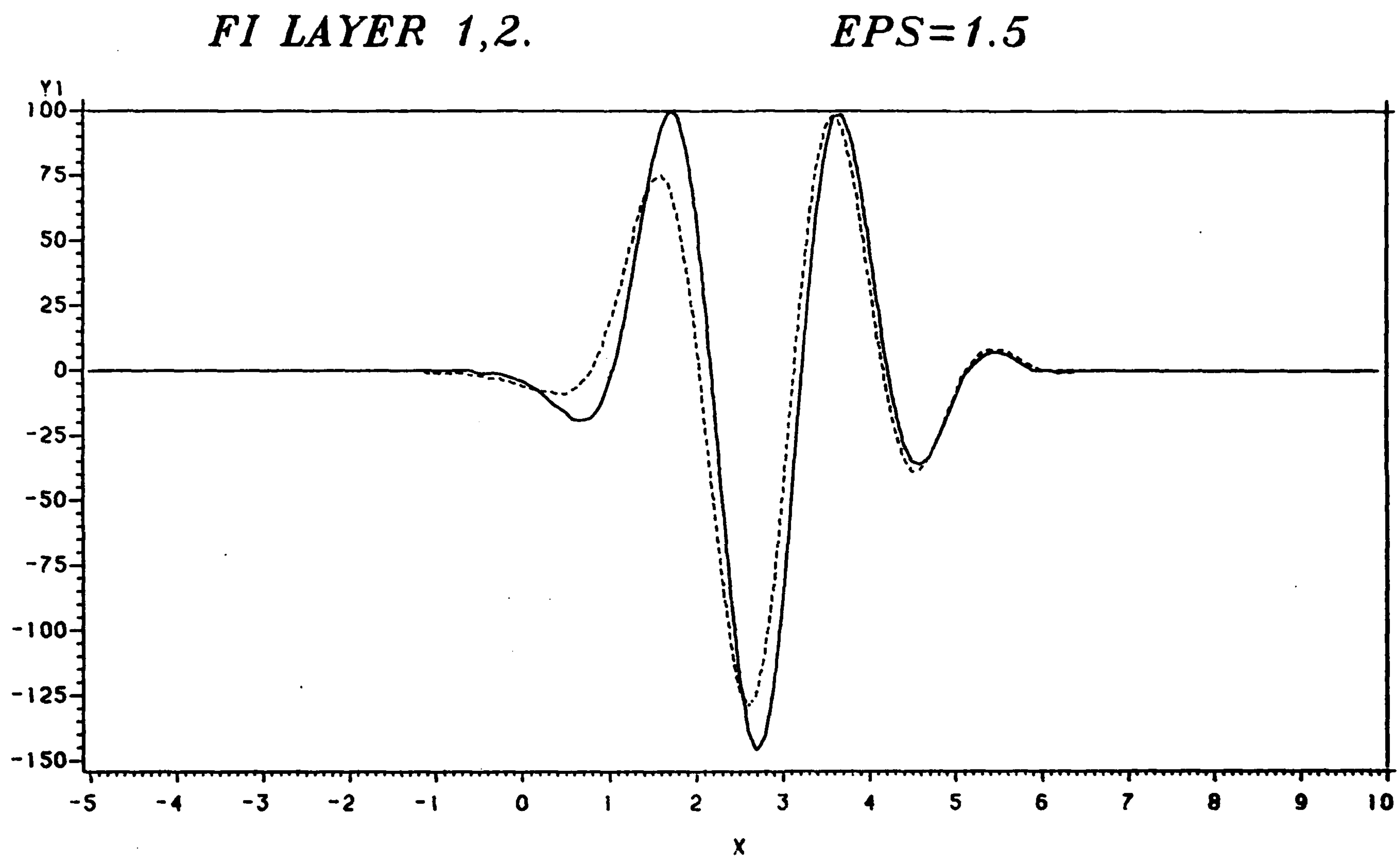
60
-30
90

Fig. 6b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-1
0
1
2
3
4
5
6
7
8
9
10
X

Fig. 7